TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 2, Pages 491–535
S 0002-9947(01)02860-4
Article electronically published on September 26, 2001

QUANTITATIVE ESTIMATES OF UNIQUE CONTINUATION FOR PARABOLIC EQUATIONS AND INVERSE INITIAL-BOUNDARY VALUE PROBLEMS WITH UNKNOWN BOUNDARIES

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Abstract. In this paper we obtain quantitative estimates of strong unique continuation for solutions to parabolic equations. We apply these results to prove stability estimates of logarithmic type for an inverse problem consisting in the determination of unknown portions of the boundary of a domain Ω in $\mathbb{R}^n$, from the knowledge of overdetermined boundary data for parabolic boundary value problems.

1. Introduction

In this paper we prove quantitative estimates of strong unique continuation for solutions to the parabolic equation

$$u_t(x,t) - \operatorname{div}(\kappa(x)\nabla u(x,t)) = 0, \quad \text{in } \Omega \times (0,T], \tag{1.1}$$

where T is a positive number, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with sufficiently smooth boundary, $\kappa(x) = \{\kappa_{ij}(x)\}_{i,j=1}^n$ is a Lipschitz continuous matrix valued function satisfying a uniform ellipticity condition in Ω (see Part I, below). These estimates provide the main tools we use to obtain stability estimates for an inverse initial-boundary value problem concerning the determination of unknown boundaries (see Part II, below).

Part I (*Quantitative Estimates of Unique Continuation for Parabolic Equations*). We prove the following kinds of quantitative estimates of strong unique continuation for solutions to the parabolic equation (1.1):

a) Three spheres inequalities in the interior on the characteristic planes $t = t_0$, that is, roughly speaking,

$$\|u(\cdot,t_0)\|_{L^2(\Delta_\rho)} \leq C\|u(\cdot,t_0)\|_{L^2(\Delta_r)}^{\frac{C}{\log(R/r)}} \|u(\cdot,t_0)\|_{L^2(\Delta_R\times(0,T))}^{1-\frac{C}{\log(R/r)}},$$

for every $t_0 \in (0, T/2)$ and $r < \rho < R$, where Δ_s denotes the open ball of radius s and center at 0 in $\mathbb{R}^n$. See Theorems 3.1.1 and 3.1.1′ for a precise statement. The above inequality implies strong unique continuation on the

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Received by the editors September 25, 2000.
2000 *Mathematics Subject Classification.* Primary 35R30; Secondary 35R25, 35R35.
Key words and phrases. Parabolic equations, strong unique continuation, stability, inverse problems.
Work supported in part by MURST.

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characteristic planes $t = t_0$ (see Corollary 3.1.7) and three cylinders inequalities in the interior (see Theorems 3.1.1 and 3.1.1′).

b) Three spheres inequalities at the boundary on the characteristic planes $t = t_0$ and three cylinders inequalities at the boundary (see Theorems 3.2.1 and 3.2.1′).
c) Stability estimates of continuation from Cauchy data on time-like surfaces (see Theorem 3.3.1).
d) Stability estimates of continuation from the interior (see Proposition 5.5).

The main tool which we use to derive all of these estimates is the so-called *elliptic continuation of solutions to parabolic equations* [Lan-O], which can be traced back to the pioneering work by Yamabe [Y], who introduced this technique in 1959 to prove weak unique continuation properties for solutions to (1.1), when $\kappa \in C^3$ (see also [ItY]).

Roughly speaking, the above mentioned elliptic continuation technique consists in the following idea: fixing $t_0 \in (0, T)$, a solution $u(x, t)$ to the parabolic equation (1.1) can be continued for values of y, with $|y| < \delta$, in such a way that $u(x, t_0, y)$ satisfies an elliptic equation in x and y. In this way, many properties of solutions of elliptic equations can be transferred to solutions of parabolic equations.

This technique was used again for parabolic equations of order $2m$ by Landis and Oleinik [Lan-O] in 1974, to prove three spheres inequalities on the characteristic planes and strong unique continuation for solutions to (1.1) on these planes, under very strong regularity assumptions on κ, and, more recently, by Lin [Li] in 1990, to prove strong unique continuation for solutions to (1.1), assuming κ Lipschitz continuous.

We exploit the above described elliptic continuation technique and quantitative estimates of strong unique continuation in the interior for elliptic equation (see [Lan], [GL], [K]) to find estimates of type a).

Assuming that the boundary $\partial\Omega$ is of class $C^{1,1}$, we prove estimates of type b), by combining estimates of type a) with flattening of the boundary and reflection of the solution, elaborating on arguments contained in [AE].

In the elliptic continuation technique, analyticity properties with respect to y of solutions to the elliptic equation

$$\operatorname{div}(\kappa(x)\nabla w(x, y)) + w_{yy}(x, y) = 0, \tag{1.2}$$

which is associated to (1.1) in the way described above, are used. We obtain these properties and the relative bounds on the derivatives of any order with respect to y by adapting to (1.2) the Morrey-Nirenberg technique (see [M]). Furthermore, we need to find some stability estimates of continuation from Cauchy data on the plane $y = 0$ for solutions to (1.2). We succeed in obtaining these estimates by exploiting hyperbolic continuation for solutions to (1.2), energy estimates and analytic continuation estimates.

Concerning estimates of type c), we recall that in [Is1] and in [P] estimates of this kind are proved when $\kappa \in C^1$, see also [LavRS] (in the quoted papers κ may also depend on t). When κ only depends on x, we obtain the required estimates assuming that κ is merely Lipschitz continuous, by using an extension in H^2 of the Cauchy data and the three spheres inequalities in the interior.

To obtain estimates of type d) we again combine estimates of type a) with the arguments used to prove Proposition 4.3 in [AlBRV].

Part II (*Determining Boundaries in Inverse Conduction Problems*). We deal with an inverse problem which arises in thermal imaging. This is a technique of non-destructive testing, to detect unknown corroded portions of the boundary or cavities in material objects, by measurements of temperature on an accessible portion of the surface. We refer to [BryC1], [BryC2] and references therein for more details and applications. More precisely, let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$, and let us assume that an insulated portion I of $\partial\Omega$ is not known. The thermal imaging technique consists in giving a heat flux input on an accessible portion A of $\partial\Omega$ and measuring the temperature on A. Let us assume that $\kappa(x) = \{\kappa_{ij}(x)\}_{i,j=1}^n$ is the known symmetric thermal conductivity matrix, satisfying a uniform ellipticity condition, that f is a known function on Ω and that g is a nontrivial function on $\partial\Omega\times(0,T)$ such that $\mathrm{supp} g \subset A$. Then the temperature u in Ω satisfies the following initial-boundary value problem of Neumann type:

$$u_t(x,t) - \mathrm{div}(\kappa(x)\nabla u(x,t)) = 0, \quad \text{in } \Omega \times (0,T], \tag{1.3a}$$

$$u = f, \quad \text{on } \bar{\Omega} \times \{0\}, \tag{1.3b}$$

$$\kappa\nabla u \cdot \nu = g, \quad \text{on } \partial\Omega \times [0,T], \tag{1.3c}$$

where ν denotes the outer unit normal to Ω. It is well known that (1.3) has a unique solution.

Given an open subset Σ of the boundary of Ω which is contained in A, we consider the inverse problem of determining I from the knowledge of u on $\Sigma \times [0,T]$.

We recall that uniqueness results for this problem have been proved by Isakov in [Is2] when $f \equiv 0$, κ also depends on t, and other terms of lower order appear in equation (1.3a). In [BryC1] Bryan and Caudill gave a counterexample showing that uniqueness may fail when $f \not\equiv 0$.

Let us also recall that in [C] the problem of determining I from infinitely many measurements at a given time t_0 is studied.

In this paper we are mainly interested in studying the stability, that is, the continuous dependence of I on the Cauchy data u, $\kappa\nabla u \cdot \nu$ on $\Sigma \times [0,T]$.

There is clear evidence that this inverse problem is severely ill-posed. In fact, in order to determine the unknown boundary I it seems necessary to determine the interior values of u from the Cauchy data on $\Sigma \times (0,T)$ up to I. Therefore it is reasonable to expect a weak rate of stability under a priori information on the unknown boundary portion I.

In 1997 Vessella ([V]) considered the case in which $\kappa \equiv Id$, the temperature is prescribed and the heat flux is measured. He proved logarithmic stability estimates, assuming that the prescribed temperature on $\partial\Omega$ is monotone with respect to time, by using analytic continuation techniques and properties of solutions to parabolic equations of constant sign.

Here, we prove logarithmic stability estimates under some a priori information on the domain, on I and on the oscillation character of g. The proof of the stability result has the same structure as that in [AlBRV], and consists of two main steps. As a first step, we prove a rough estimate for the Hausdorff distance between the domains Ω_1, Ω_2 corresponding to the solutions u_1, u_2 whose boundary measurements are known with error, by using the estimates of type a)-d) described in Part I of this Introduction. As a second step, we employ in a more refined way the above mentioned estimates and a geometric lemma (Proposition 5.6), which has been proved in [AlBRV], obtaining the logarithmic stability estimate.

Indeed, in the elliptic case, counterexamples by Alessandrini [Al] and Alessandrini and Rondi [AlR] show that logarithmic stability is best possible. This suggests that also in the parabolic case stability estimates better than logarithmic cannot be expected.

An interesting open problem is to find logarithmic stability estimates in the case in which κ depends also on t. In fact, in this general case some estimates of unique continuation are available (see [Is1], [LeP], [P], [LavRS]), but neither three spheres inequalities on the characteristic planes at the boundary nor three cylinders inequalities at the boundary are known. On the other side, these estimates at the boundary are the main tools we use in the proof of our stability result.

Connected to the inverse problem presented above is the question of uniqueness and stability when more general boundary conditions are prescribed on the unknown portion of the boundary, like, for instance,

$$\kappa\nabla u\cdot\nu+\alpha u=0,\quad \text{on } \partial\Omega\times[0,T],$$

where $\alpha\geq 0$ is a known function.

The plan of the paper is as follows.

In Section 2 we introduce some notation and definitions.

In Section 3, which we have subdivided into three subsections, we prove some quantitative estimates of strong unique continuation for solutions to (1.1), namely: three spheres inequalities in the interior (Theorem 3.1.1 and Theorem 3.1.1′), strong unique continuation on characteristic planes (Corollary 3.1.7), three spheres inequalities at the boundary (Theorem 3.2.1 and Theorem 3.2.1′), stability estimates of continuation from Cauchy data on time-like surfaces (Theorem 3.3.1), and the stability estimates of continuation from Cauchy data on the plane $y=0$ for solutions to (1.2) (Proposition 3.1.4).

In Section 4 we state the stability result for the inverse problem described in Part II (Theorem 4.1).

Section 5 contains the proof of Theorem 4.1, which is obtained through a sequence of Propositions. In particular, Proposition 5.5 provides estimates of type d).

Section 6 contains the proof of Propositions 5.1-5.5.

Finally, in Appendix A we have collected some interpolation and traces inequalities which we use throughout the paper.

2. Notation and Definitions

Let us introduce some notation.

We shall fix the space dimension $n\geq 3$ throughout the paper. Therefore we shall omit the dependence of the various quantities on n.

We shall use the letter c to denote absolute constants, and the letters C, $\tilde{C}$ to denote constants depending on some a priori data. The value of the constants may change from line to line, but we have specified their dependence everywhere they appear.

We shall identify $\mathbb{R}^2$ and $\mathbb{C}$.

We shall denote by $B_r(a)$ ($\Delta_r(a)$, $\Delta'_r(a)$, $D_r(a)$, respectively) the open ball in $\mathbb{R}^{n+1}$ ($\mathbb{R}^n$, $\mathbb{R}^{n-1}$, $\mathbb{C}$ respectively) centered at a, of radius r. Sometimes we shall write for brevity B_r, Δ_r, Δ'_r, D_r instead of $B_r(0)$, $\Delta_r(0)$, $\Delta'_r(0)$, $D_r(0)$, respectively.

We shall denote $B_r^+ = \{(x_1,...,x_n,y) \in B_r \text{ s.t. } y > 0\}$, $\Delta_r^+ = \{(x_1,...,x_n) \in \Delta_r \text{ s.t. } x_n > 0\}$. When dealing with $n+1$ variables $(x,\cdot)$, where $x=(x_1,...,x_n)$, we shall denote $\nabla = \nabla_x$, $\text{div} = \text{div}_x$, $D^2 = D_x^2$. Sometimes, for brevity, we shall write $\partial_y^k u$ instead of $\frac{\partial^k u}{\partial y^k}$, u_y instead of $\frac{\partial u}{\partial y}$ and u_{yy} instead of $\frac{\partial^2 u}{\partial y^2}$.

Given an open set $\Omega \subset \mathbb{R}^n$ and $r>0$, we shall denote

$$\Omega_r = \{x \in \Omega \text{ s.t. } \text{dist}(x, \partial\Omega) > r\}.$$

When representing a boundary locally as a graph, it will be convenient to use the following notation. For every $x \in \mathbb{R}^n$ we shall set $x=(x', x_n)$, where $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$.

Definition 2.1. Let Ω be a bounded domain in $\mathbb{R}^n$. We shall say that a portion S of $\partial\Omega$ is of *class* $C^{1,1}$ *with constants* R_0, $E>0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P=0$ and

$$\Omega \cap \Delta_{R_0} = \{x \in \Delta_{R_0} \text{ s.t. } x_n > \varphi(x')\},$$

where φ is a $C^{1,1}$ function on $\Delta'_{R_0} \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$\|\varphi\|_{C^{1,1}(\Delta'_{R_0})} \leq ER_0.$$

Definition 2.2. Let Ω be a bounded domain in $\mathbb{R}^n$. We shall say that a portion S of $\partial\Omega$ is of *Lipschitz class with constants* R_0, $E>0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P=0$ and

$$\Omega \cap \Delta_{R_0} = \{x \in \Delta_{R_0} \text{ s.t. } x_n > \varphi(x')\},$$

where φ is a Lipschitz continuous function on $\Delta'_{R_0} \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = 0$$

and

$$\|\varphi\|_{C^{0,1}(\Delta'_{R_0})} \leq ER_0.$$

Remark 2.1. We have chosen to normalize all norms in such a way that their terms are dimensionally homogeneous, and coincide with the standard definition when $R_0 = 1$ and $T=1$. For instance, for any $\varphi \in C^{1,1}(\Delta'_{R_0})$, we set

$$\|\varphi\|_{C^{1,1}(\Delta'_{R_0})} = \|\varphi\|_{L^\infty(\Delta'_{R_0})} + R_0\|\nabla\varphi\|_{L^\infty(\Delta'_{R_0})} + R_0^2\|D^2\varphi\|_{L^\infty(\Delta'_{R_0})},$$

and, for any $\varphi \in C^{0,1}(\Delta'_{R_0})$, we set

$$\|\varphi\|_{C^{0,1}(\Delta'_{R_0})} = \|\varphi\|_{L^\infty(\Delta'_{R_0})} + R_0|\varphi|_{1,\Delta'_{R_0}}.$$

Similarly, for any $u \in H^{2,1}(\Omega \times (0,T))$, we set

$$\|u\|^2_{H^{2,1}(\Omega\times(0,T))} = \int_{\Omega\times(0,T)} \left(u^2 + R_0^2|\nabla u|^2 + R_0^4|D^2u|^2 + T^2u_t^2\right) dxdt,$$

and so on for boundary and trace norms such as $\|\cdot\|_{H^{3/2,3/4}(S_T)}$, $\|\cdot\|_{H^{1/2,1/4}(S_T)}$, where we denote

$$S_T = \partial\Omega \times (0,T).$$

3. Quantitative estimates of strong unique continuation

We subdivide this section into three subsections.

In Subsection 3.1 we exploit the elliptic continuation technique to obtain three spheres inequalities in the interior and strong unique continuation on characteristic planes. First, we prove a form of the three spheres inequality in the interior suitable for the applications of Section 4, under the additional assumption that the solution vanishes at time $t = 0$ (Theorem 3.1.1). Next, by a slight modification of the proof of Theorem 3.1.1, we prove the three spheres inequality in the interior also for solutions taking general initial data (Theorem 3.1.1′). As a consequence of Theorem 3.1.1′, we derive strong unique continuation on characteristic planes (Corollary 3.1.7).

In Subsection 3.2, we prove the three spheres inequality at the boundary for solutions taking zero initial values (Theorem 3.2.1) and for solutions taking general initial values (Theorem 3.2.1′).

In Subsection 3.3 we prove a stability result for the Cauchy problem for parabolic equations on time-like surfaces (Theorem 3.3.1).

Subsection 3.1 (Three Spheres Inequalities and Three Cylinders Inequalities in the Interior). We are given positive constants T, R_0, R, λ, λ_1, Λ and Λ_1, satisfying $\lambda \geq 1$, $\lambda_1 \geq 1$, $2R \leq R_0$.

Let us denote by c_P the absolute constant from the following Poincaré inequality:

$$\int_{\Delta_{2R}} f^2(x)dx \leq c_P R^2 \int_{\Delta_{2R}} |\nabla f(x)|^2 dx, \quad \text{for every } f \in H^1_0(\Delta_{2R}),$$

where we recall that $c_P \leq 4$ (see [GT]) and that $c_P = \frac{4}{k_0^2}$, where k_0 is the smallest positive root of the Bessel function of first kind $J_{\frac{n-2}{2}}$ (see [CH]).

Let $\kappa(x)$ be a given function from Δ_{2R}, with values $n \times n$ symmetric matrices, satisfying the following conditions:

(3.1.1a)
$$\lambda^{-1}|\xi|^2 \leq \kappa(x)\xi \cdot \xi \leq \lambda|\xi|^2, \quad \text{for every } x \in \Delta_{2R} \text{ and } \xi \in \mathbb{R}^n, \quad (\textit{ellipticity})$$

(3.1.1b)
$$|\kappa(x) - \kappa(y)| \leq \frac{\Lambda}{R_0}|x - y|, \quad \text{for every } x, y \in \Delta_{2R}. \quad (\textit{Lipschitz continuity})$$

Let $b(x)$ be a given function from Δ_{2R} satisfying the following conditions:

(3.1.2a)
$$\lambda_1^{-1} \leq b(x) \leq \lambda_1, \quad \text{for every } x \in \Delta_{2R},$$

(3.1.2b)
$$|b(x) - b(y)| \leq \frac{\Lambda_1}{R_0}|x - y|, \quad \text{for every } x, y \in \Delta_{2R}.$$

Let

(3.1.3)
$$\lambda_2 = \max\{\lambda, \lambda_1\},$$

(3.1.4)
$$\Lambda_2 = \max\{\Lambda, \Lambda_1\}.$$

Let $u(x,t) \in H^{2,1}(\Delta_{2R} \times (0,T))$ be a solution to the parabolic equation

(3.1.5a)
$$b(x)u_t(x,t) - \mathrm{div}(\kappa(x)\nabla u(x,t)) = 0, \quad \text{in } \Delta_{2R} \times (0,T],$$

satisfying

$$u(\cdot,0)=0, \quad \text{on } \Delta_{2R}\times\{0\}, \tag{3.1.5b}$$

and let

$$H=\sup_{t\in[0,T]}\|u(\cdot,t)\|_{H^1(\Delta_{2R})}, \tag{3.1.6}$$

where

$$\|u(\cdot,t)\|^2_{H^1(\Delta_{2R})}=\int_{\Delta_{2R}}(u^2(x,t)+R_0^2|\nabla u(x,t)|^2)dx.$$

Theorem 3.1.1. (Three Spheres Inequalities and Three Cylinders Inequalities in the Interior). *Let $u\in H^{2,1}(\Delta_{2R}\times(0,T))$ be a solution to* (3.1.5) *and let* (3.1.1) *and* (3.1.2) *be satisfied. For every r_1, r_2, r_3 with $0<r_1<r_2<(4\max\{\sqrt{2},\lambda_2\})^{-1}r_3 < r_3\leq\bar{\theta}R$, we have*

$$\int_{\Delta_{r_2}}u^2(x,t_0)dx$$

(3.1.7a)

$$\leq \tilde{C}\frac{r_3}{r_3-r_2}\left(\frac{r_3}{r_2}\right)^C\left(\left(1+\frac{T^2}{R^4}\right)H^2\right)^{1-\bar{\beta}\gamma'}\left(\int_{\Delta_{r_1}}u^2(x,t_0)dx\right)^{\bar{\beta}\gamma'},\forall t_0\in(0,\frac{T}{2}),$$

$$\int_0^{\frac{T}{2}}\int_{\Delta_{r_2}}u^2(x,t)dxdt$$

(3.1.7b)

$$\leq \tilde{C}\frac{r_3}{r_3-r_2}\left(\frac{r_3}{r_2}\right)^C\left(T\left(1+\frac{T^2}{R^4}\right)H^2\right)^{1-\bar{\beta}\gamma'}\left(\int_0^{\frac{T}{2}}\int_{\Delta_{r_1}}u^2(x,t)dxdt\right)^{\bar{\beta}\gamma'},$$

where

$$\gamma'=\frac{\log\left(\frac{1}{2}+\frac{r_3}{2\lambda_2 r_2'}\right)}{\log\left(\frac{1}{2}+\frac{r_3}{2\lambda_2 r_2'}\right)+C\log\frac{2\lambda_2 r_2'}{ar_1}}, \tag{3.1.8a}$$

$$r_2'=\frac{4\lambda_2-2}{4\lambda_2-1}r_2+\frac{1}{4\lambda_2-1}r_3, \tag{3.1.8b}$$

$$a=\frac{3}{4\sqrt{2}e\pi\lambda_2}, \tag{3.1.8c}$$

$\bar{\beta}$, $0<\bar{\beta}<1$, depends on λ_2 only, $\bar{\theta}$, $0<\bar{\theta}<1$, and C depend on λ_2 and Λ_2 only and $\tilde{C}$ depends on λ_2, Λ_2 and $\frac{R_0^2}{T}$ only.

Remark 3.1.2. i) Here, we study the more general equation (3.1.5a) instead of (1.1), since in our proof of the three spheres inequalities at the boundary (Theorem 3.2.1) we shall need the estimates (3.1.7) for solutions to equations of the form (3.1.5a) with a coefficient b satisfying (3.1.2).

ii) Actually, a more correct terminology for inequality (3.1.7a) should be *two spheres and one cylinder inequality* (see the definition (3.1.6) of H), but we call it *three spheres inequality* for historical reasons (see [Lan-O]).

iii) We have expressed the three cylinders inequality (3.1.7b) in terms of the $L^\infty - L^2$ norm over the cylinder $\Delta_{2R}\times(0,T)$ (see the definition (3.1.6) of H). Let us notice, however, that also a more symmetric inequality, involving the L^2 norm over $\Delta_{2R}\times(0,T)$, could be derived similarly.

In order to prove Theorem 3.1.1, we prove some preliminary lemmas concerning the technique of elliptic continuation (see also [Li]) and a stability estimate of continuation from Cauchy data on the plane $y = 0$ for solutions to the elliptic equation (1.2) (Proposition 3.1.4).

Let us start by fixing $t_0 \in (0,T)$ and by considering the weak solution $\tilde{u}$ to the following initial-boundary value parabolic problem:

$$b(x)\tilde{u}_t(x,t) - \mathrm{div}(\kappa(x)\nabla\tilde{u}(x,t)) = 0, \quad \text{in } \Delta_{2R}\times(0,+\infty), \tag{3.1.9a}$$

$$\tilde{u} = h, \quad \text{on } \partial\Delta_{2R}\times[0,+\infty), \tag{3.1.9b}$$

$$\tilde{u} = 0, \quad \text{on } \Delta_{2R}\times\{0\}, \tag{3.1.9c}$$

where h is the extension by 0 of $\eta(t)u(x,t)$, and where $\eta\in C^2([0,1])$ is a cut-off function such that $\eta\equiv 1$ in $[0,t_0]$, $\eta\equiv 0$ in $[T,+\infty)$ and $|\eta'|\le \frac{c}{T-t_0}$.

It is evident that

$$\tilde{u}(\cdot,t_0) = u(\cdot,t_0), \qquad \text{in } \Delta_{2R}.$$

Let us denote

$$a_R = (\lambda\lambda_1 c_P R^2)^{-1}, \tag{3.1.10}$$

$$q = \frac{T}{T-t_0}. \tag{3.1.11}$$

Lemma 3.1.2. *Let u, $\tilde{u}$ be as above. We have*

$$\|\tilde{u}(\cdot,t)\|^2_{H^1(\Delta_{2R})} \le cC_1^2H^2\exp(-2a_R(t-T)_+), \tag{3.1.12}$$

where

$$C_1^2 = \lambda_2^2 q\left(\frac{R_0^2}{T-t_0} + e^{cq}\right). \tag{3.1.13}$$

Proof of Lemma 3.1.2. Let

$$v = \tilde{u} - \eta u,$$

where η is the cut-off function introduced above. We have that v satisfies

$$b(x)v_t(x,t) - \mathrm{div}(\kappa(x)\nabla v(x,t)) = -b(x)\eta'(t)u(x,t), \quad \text{in } \Delta_{2R}\times(0,T], \tag{3.1.14a}$$

$$v = 0, \quad \text{on } \partial\Delta_{2R}\times[0,T], \tag{3.1.14b}$$

$$v = 0, \quad \text{on } \Delta_{2R}\times\{0\}. \tag{3.1.14c}$$

Multiplying equation (3.1.14a) by v and integrating over $\Delta_{2R}\times[0,t]$, we obtain, for $t\in[0,T]$,

$$\int_{\Delta_{2R}} b(x)v^2(x,t)dx \le c\lambda_1 qH^2 + \left(\frac{c}{T-t_0}\right)\int_0^t d\tau\int_{\Delta_{2R}} b(x)v^2(x,\tau)dx. \tag{3.1.15}$$

By applying the Peano-Gronwall inequality to (3.1.15), we have

$$\int_{\Delta_{2R}} b(x)v^2(x,t)dx \le c\lambda_1 q e^{cq} H^2. \tag{3.1.16}$$

Choosing $t = T$ in (3.1.16) and recalling that $v(\cdot, T) = \tilde{u}(\cdot, T)$, we obtain

$$\int_{\Delta_{2R}} b(x)\tilde{u}^2(x,T)dx \le c\lambda_1 q e^{cq} H^2. \tag{3.1.17}$$

Similarly, multiplying equation (3.1.14a) by v_t and integrating over $\Delta_{2R} \times [0,t]$, we obtain, for $t \in [0,T]$,

$$\int_{\Delta_{2R}} \kappa(x)\nabla v(x,t) \cdot \nabla v(x,t)dx \le c\lambda_1 \frac{t}{(T-t_0)^2} H^2, \tag{3.1.18}$$

$$\int_{\Delta_{2R}} \kappa(x)\nabla \tilde{u}(x,T) \cdot \nabla \tilde{u}(x,T)dx \le c\lambda_1 \frac{T}{(T-t_0)^2} H^2. \tag{3.1.19}$$

Let us denote by μ_k, with $\mu_1 > \mu_2 \ge ... \ge \mu_k \ge ...$, the negative eigenvalues associated to the problem

$$\frac{1}{b(x)}\mathrm{div}(\kappa(x)\nabla\varphi(x)) = \mu\varphi(x), \qquad \text{in } \Delta_{2R}, \tag{3.1.20a}$$

$$\varphi = 0, \qquad \text{on } \partial\Delta_{2R}, \tag{3.1.20b}$$

and by φ_k the corresponding eigenfunctions normalized by

$$\int_{\Delta_{2R}} b(x)\varphi_k^2(x)dx = 1.$$

We have

$$0 > -a_R \ge \mu_1 > \mu_2 \ge ... \ge \mu_k \ge .... \tag{3.1.21}$$

Since $\tilde{u} = 0$ on $\partial\Delta_{2R} \times [T, +\infty)$, we have

$$\tilde{u}(x,t) = \sum_{k=1}^{\infty} \beta_k \varphi_k(x) e^{\mu_k(t-T)}, \qquad \text{for every } t \ge T, \tag{3.1.22a}$$

where

$$\beta_k = \int_{\Delta_{2R}} b(x)\varphi_k(x)\tilde{u}(x,T)dx. \tag{3.1.22b}$$

Recalling (3.1.17), we have

$$\int_{\Delta_{2R}} b(x)\tilde{u}^2(x,t)dx = \sum_{k=1}^{\infty} \beta_k^2 e^{2\mu_k(t-T)} \le e^{-2a_R(t-T)} \int_{\Delta_{2R}} b(x)\tilde{u}^2(x,T)dx$$

$$\le c\lambda_1 q e^{cq} H^2 e^{-2a_R(t-T)}, \quad \text{for every } t \ge T. \tag{3.1.23}$$

Moreover,

$$\int_{\Delta_{2R}} \kappa(x)\nabla\tilde{u}(x,t) \cdot \nabla\tilde{u}(x,t)dx = -\int_{\Delta_{2R}} b(x)\tilde{u}(x,t)\tilde{u}_t(x,t)dx \tag{3.1.24}$$

$$= \sum_{k=1}^{\infty} |\mu_k|\beta_k^2 e^{2\mu_k(t-T)}.$$

Choosing $t = T$ in (3.1.24) and using (3.1.19), we can estimate

$$\sum_{k=1}^{\infty} |\mu_k|\beta_k^2 \le c\lambda_1 \frac{T}{(T-t_0)^2} H^2. \tag{3.1.25}$$

From (3.1.24) and (3.1.25) we have

$$R_0^2 \int_{\Delta_{2R}} |\nabla \tilde{u}(x,t)|^2 dx \le c\lambda\lambda_1 \frac{TR_0^2}{(T-t_0)^2} H^2 e^{-2a_R(t-T)} \quad \text{for every } t \ge T. \tag{3.1.26}$$

From (3.1.23) and (3.1.26) we obtain

(3.1.27)
$$\|\tilde{u}(\cdot,t)\|^2_{H^1(\Delta_{2R})} \le c\lambda_1\lambda_2 q \left(e^{cq} + \frac{R_0^2}{(T-t_0)} \right) H^2 e^{-2a_R(t-T)} \quad \text{for every } t \ge T.$$

On the other hand, from (3.1.16) and (3.1.18), recalling that $\tilde{u} = v + \eta u$, we have

$$\|\tilde{u}(\cdot,t)\|^2_{H^1(\Delta_{2R})} \le c\lambda_1\lambda_2 q \left(e^{cq} + \frac{R_0^2}{(T-t_0)} \right) H^2 \quad \text{for every } t \le T, \tag{3.1.28}$$

so that (3.1.12) follows from (3.1.27) and (3.1.28). □

Let us still denote by $\tilde{u}$ the extension by 0 of $\tilde{u}$ to $\Delta_{2R} \times \mathbb{R}$, and let us consider the Fourier transform of $\tilde{u}$ with respect to the variable t,

$$\hat{u}(x,\mu) = \int_{-\infty}^{+\infty} e^{-\mu i t} \tilde{u}(x,t) dt = \int_0^{+\infty} e^{-\mu i t} \tilde{u}(x,t) dt, \quad \text{for every } \mu \in \mathbb{R}. \tag{3.1.29}$$

We have that $\hat{u}$ satisfies

$$b(x) i\mu \hat{u}(x,\mu) - \mathrm{div}(\kappa(x)\nabla \hat{u}(x,\mu)) = 0, \quad \text{in } \Delta_{2R} \times \mathbb{R}. \tag{3.1.30}$$

Lemma 3.1.3. *Let u, $\tilde{u}$, $\hat{u}$ be as above. We have*

$$\|\hat{u}(\cdot,\mu)\|_{L^2(\Delta_R)} \le cC_1\lambda_2 H e^{-\sqrt{|\mu|}\delta} (T + \frac{1}{a_R}), \tag{3.1.31}$$

where

$$\delta = \frac{R}{4e\pi\lambda_2}. \tag{3.1.32}$$

Proof of Lemma 3.1.3. Let us denote, for every μ, $\xi \in \mathbb{R}$, $x \in \Delta_{2R}$,

$$v(x,\xi;\mu) = e^{i\sqrt{|\mu|}\xi} \hat{u}(x,\mu).$$

For every $\mu \in \mathbb{R}\backslash\{0\}$, the function $v = v(\cdot,\cdot;\mu)$ solves the uniformly elliptic equation

$$\mathrm{div}(\kappa\nabla v) + ib\,\mathrm{sgn}(\mu) \frac{\partial^2 v}{\partial \xi^2} = 0, \quad \text{in } \Delta_{2R} \times \mathbb{R}. \tag{3.1.33}$$

Let us denote $a_j = 2 - \frac{j}{k}$, for every $j \in \{0, 1, ..., k\}$, and $k \in \mathbb{N}$. Moreover, let

$$h_j(s) = \begin{cases} 0 & \text{if } |s| > a_j, \\ \frac{1}{2}\left(1 + \cos\left(\frac{\pi(a_{j+1}-s)}{a_{j+1}-a_j}\right)\right) & \text{if } a_{j+1} \le |s| \le a_j, \\ 1 & \text{if } |s| < a_{j+1}, \end{cases}$$

and

$$v_j = \frac{\partial^j v}{\partial \xi^j}.$$

We have that v_j solves the equation

$$\text{div}(\kappa \nabla v_j) + ib\text{sgn}(\mu)\frac{\partial^2 v_j}{\partial \xi^2} = 0, \quad \text{in } \Delta_{2R} \times \mathbb{R}. \tag{3.1.34}$$

Multiplying equation (3.1.34) by $\bar{v}_j \eta_j^2$, where

$$\eta_j = \eta_j(x,\xi) = h_j\left(\frac{|x|}{R}\right) h_j\left(\frac{\xi}{R}\right),$$

and integrating over $D_j = \Delta_{a_j R} \times (-a_j R, a_j R)$, we obtain

$$\left(\int_{D_j} \kappa \nabla v_j \cdot \nabla \bar{v}_j \eta_j^2 dx d\xi\right)^2 + \left(\int_{D_j} b\left|\frac{\partial v_j}{\partial \xi}\right|^2 \eta_j^2 dx d\xi\right)^2$$

$$\leq \frac{8\lambda_2^2 \pi^4 k^4}{R^4}\left(\int_{D_j} |v_j|^2 dx d\xi\right)^2. \tag{3.1.35}$$

Therefore, for every $j \in \{0, 1, ..., k\}$ we obtain

$$\int_{D_{j+1}} \left|\frac{\partial^{j+1} v}{\partial \xi^{j+1}}\right|^2 dx d\xi \leq \frac{\sqrt{8}\lambda_1 \lambda_2 \pi^2 k^2}{R^2} \int_{D_j} \left|\frac{\partial^j v}{\partial \xi^j}\right|^2 dx d\xi. \tag{3.1.36}$$

By iteration of (3.1.36) for $j = 0, ..., k-1$, we have

$$\int_{\Delta_R \times (-R,R)} \left|\frac{\partial^k v}{\partial \xi^k}\right|^2 dx d\xi \leq 4R\left(\frac{\sqrt{8}\lambda_1 \lambda_2 \pi^2 k^2}{R^2}\right)^k \int_{\Delta_{2R}} \hat{u}^2(x,\mu) dx. \tag{3.1.37}$$

Now, let us estimate the integral on the right hand side of (3.1.37). By (3.1.12), we obtain

$$\|\hat{u}(\cdot,\mu)\|_{L^2(\Delta_{2R})} \leq \int_0^{+\infty} \|\tilde{u}(\cdot,t)\|_{L^2(\Delta_{2R})} dt \leq cC_1 H\left(T + \frac{1}{a_R}\right). \tag{3.1.38}$$

Therefore, by (3.1.37), we have that, for every $\mu \in \mathbb{R} \setminus \{0\}$ and for every $k \in \mathbb{N}$,

$$\int_{\Delta_R \times (-R,R)} \left|\frac{\partial^k v}{\partial \xi^k}\right|^2 dx d\xi \leq cR\left(T + \frac{1}{a_R}\right)^2 C_1^2 H^2 \left(\frac{\sqrt{8}\lambda_1 \lambda_2 \pi^2 k^2}{R^2}\right)^k. \tag{3.1.39}$$

For fixed $\mu \in \mathbb{R} \setminus \{0\}$ and $\varphi \in L^2(\Delta_R, \mathbb{C})$, let us denote

$$F(\xi) = \int_{\Delta_R} v(x,\xi;\mu)\bar{\varphi}(x) dx.$$

By the interpolation inequality (A.1) and by the inequality (3.1.39) we have

$$|F^{(k)}(\xi)| \leq \frac{cC_1 H}{R^k}\left(T + \frac{1}{a_R}\right)(2\lambda_2 \pi(k+1))^{k+1} \|\varphi\|_{L^2(\Delta_R)}. \tag{3.1.40}$$

By using inequality (3.1.40) for every $k \in \mathbb{N}$ and the power series of F at any point ξ_0 such that $\Re e \xi_0 \in (-R,R)$, $\Im m \xi_0 = 0$, we have that the function F can be analitycally extended to the rectangle $\{\xi \in \mathbb{C} \text{ s.t. } \Re e \xi \in (-R,R), \Im m \xi \in (-\bar{\rho}, \bar{\rho})\}$, where $\bar{\rho} = \frac{R}{2e\pi\lambda_2}$. We continue to denote by F the analytic extension of F. In particular, choosing $\xi_0 = 0$, we obtain the estimate

$$|F(-i\delta)| \leq cC_1\lambda_2 H\left(T + \frac{1}{a_R}\right)\|\varphi\|_{L^2(\Delta_R)}, \tag{3.1.41}$$

where δ is given by (3.1.32). On the other side, by the definition of v, we have

$$F(-i\delta) = \int_{\Delta_R} e^{\sqrt{|\mu|}\delta}\hat{u}(x,\mu)\bar{\varphi}(x)dx, \tag{3.1.42}$$

so that we obtain (3.1.31) from (3.1.41). □

Estimate (3.1.31) allows us to define, for $x \in \Delta_R$, $|y| < \sqrt{2}\delta$, the function

$$w(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it_0\mu}\hat{u}(x,\mu)\cosh\sqrt{-i\mu}\; yd\mu, \tag{3.1.43}$$

where

$$\sqrt{-i\mu} = |\mu|^{\frac{1}{2}}e^{-\frac{\pi}{4}i\operatorname{sgn}(\mu)}. \tag{3.1.44}$$

We have that $w \in H^1_{loc}(\Delta_R \times (-\sqrt{2}\delta, \sqrt{2}\delta))$ is a solution to the following elliptic equation:

$$\operatorname{div}(\kappa(x)\nabla w(x,y)) + b(x)w_{yy}(x,y) = 0, \quad \text{in } \Delta_R \times (-\sqrt{2}\delta, +\sqrt{2}\delta), \tag{3.1.45a}$$

and satisfies the following conditions:

$$w(x,0) = u(x,t_0), \quad \text{in } \Delta_R, \tag{3.1.45b}$$

$$w_y(x,0) = 0, \quad \text{in } \Delta_R. \tag{3.1.45c}$$

Let us notice that w is even with respect to the y variable.

Now, let us state and prove the following stability estimate for solutions to the Cauchy problem for elliptic equations.

Proposition 3.1.4. (Stability Estimates of Continuation from Cauchy Data on the Plane $y = 0$ for Solutions to (1.2)). *Let L and ρ be positive numbers such that $0 < \rho < R$. Let w be the solution to the following Cauchy problem*:

$$\operatorname{div}(\kappa(x)\nabla w(x,y)) + b(x)w_{yy}(x,y) = 0, \quad \textit{in } \Delta_{2\rho} \times (-2L, 2L), \tag{3.1.46a}$$

$$w(x,0) = f(x), \quad \textit{in } \Delta_{2\rho}, \tag{3.1.46b}$$

$$w_y(x,0) = 0, \quad \textit{in } \Delta_{2\rho}, \tag{3.1.46c}$$

where κ satisfies (3.1.1) *and b satisfies* (3.1.2). *Let*

$$\rho_1 = \left(\pi e\lambda_2\left(\frac{1}{\rho^2} + \frac{1}{L^2}\right)^{1/2}\right)^{-1}, \tag{3.1.47}$$

$$\rho_2 = \min\left\{\rho_1, \frac{\rho}{2\sqrt{\lambda}\lambda_1}\right\}, \tag{3.1.48}$$

$$\rho_3 = \frac{1}{2}(\rho - \sqrt{\lambda}\lambda_1\rho_2). \tag{3.1.49}$$

For every $y \in \left(-\frac{3}{8}\rho_1, \frac{3}{8}\rho_1\right)$ *we have*

$$\int_{\Delta_{\rho_3}} (w_y^2(x,y) + |\nabla w(x,y)|^2) dx$$

$$\leq C\left(\frac{1}{L\rho^2}\|w\|^2_{L^2(\Delta_{2\rho}\times(-2L,2L))} + \|\nabla f\|^2_{L^2(\Delta_{\rho/2})}\right)^{1-\beta} \|\nabla f\|^{2\beta}_{L^2(\Delta_{\rho/2})}, \tag{3.1.50}$$

where β, $0<\beta<1$, *depends on* λ_2 *only and* C *depends on* λ_2 *and* $L\rho^{-1}$ *only.*

Proof of Proposition 3.1.4 (Preparation). We preface the proof with two auxiliary steps.

Step 1. *The power series*

$$\sum_{k=0}^{\infty} \frac{\partial^k w}{\partial y^k}(\cdot, 0)\frac{z^k}{k!}, \tag{3.1.51}$$

converges in $H^2(\Delta_{\frac{3}{4}\rho})$ *for every complex number* z *such that* $|z| < \rho_1$.

Proof of Step 1. Let us denote $Q_0 = \Delta_{2\rho}\times(-2L,2L)$, $Q_1 = \Delta_\rho \times (-L,L)$. By a slight modification of the arguments used to prove (3.1.37), we obtain

$$\|\partial_y^k w\|^2_{L^2(Q_1)} \leq (C_1 k^2)^k \|w\|^2_{L^2(Q_0)} \quad \text{for every } k \geq 1, \tag{3.1.52}$$

where

$$C_2 = \pi^2\lambda_2^2\left(\frac{1}{\rho^2} + \frac{1}{L^2}\right). \tag{3.1.53}$$

For any $\varphi \in L^2(\Delta_\rho)$, let

$$F(y) = \int_{\Delta_\rho} w(x,y)\varphi(x)dx.$$

By (3.1.52) and by the interpolation inequality (A.1) we obtain, for every $k \geq 1$,

$$|F^{(k)}(y)|^2 \leq \frac{C_3}{L}C_2^k(k+1)^{2(k+1)}\|w\|^2_{L^2(Q_0)}\|\varphi\|^2_{L^2(\Delta_\rho)}, \quad \text{for every } y \in (-L,L), \tag{3.1.54}$$

where C_3 depends on λ_2 and $L\rho^{-1}$ only. Therefore, for every $k \geq 1$,

$$\int_{\Delta_\rho} |\partial_y^k w(x,y)|^2 dx \leq \frac{C_3}{L}C_2^k(k+1)^{2(k+1)}\|w\|^2_{L^2(Q_0)}, \quad \text{for every } y \in (-L,L). \tag{3.1.55}$$

Let us fix $k \geq 1$ and $y \in (-L,L)$, and let us denote

$$g(\cdot) = b(\cdot)\partial_y^{k+2} w(\cdot, y), \quad \text{in } \Delta_\rho, \tag{3.1.56}$$

$$U(\cdot) = \partial_y^k w(\cdot, y), \quad \text{in } \Delta_\rho. \tag{3.1.57}$$

We have that U satisfies the equation

$$\mathrm{div}(\kappa\nabla U) = -g, \quad \text{in } \Delta_\rho. \tag{3.1.58}$$

From Caccioppoli's inequality we have

$$\|\nabla U\|^2_{L^2(\Delta_{\frac{3}{4}\rho})} \leq C\left(\rho^2\|g\|^2_{L^2(\Delta_\rho)} + \frac{1}{\rho^2}\|U\|^2_{L^2(\Delta_\rho)}\right), \tag{3.1.59}$$

where C depends on λ only. Choosing as test functions $V = (\eta^2 U_{x_i})_{x_i}$, $i = 1, ..., n$, where η is a cut-off function, we obtain, by standard H^2-estimates (see [GT], ch. 8]) and by (3.1.59),

$$\|D^2 U\|^2_{L^2(\Delta_{\frac{\rho}{2}})} \leq C\left(\frac{1}{\rho^2} + \frac{\Lambda^2}{R_0^2}\right)\left(\rho^2 \|g\|^2_{L^2(\Delta_\rho)} + \frac{1}{\rho^2}\|U\|^2_{L^2(\Delta_\rho)}\right), \tag{3.1.60}$$

where C depends on λ only. By (3.1.56), (3.1.57), (3.1.59) and (3.1.60) we have, for every $k \geq 1$,

$$\int_{\Delta_{\frac{3}{4}\rho}} |\nabla \partial_y^k w(x,y)|^2 dx \leq \frac{C_4}{L\rho^2} C_2^k (k+3)^{2(k+3)} \|w\|^2_{L^2(Q_0)}, \tag{3.1.61}$$

$$\int_{\Delta_{\frac{\rho}{2}}} |D^2 \partial_y^k w(x,y)|^2 dx \leq \frac{C_4}{L\rho^2}\left(\frac{1}{\rho^2} + \frac{\Lambda^2}{R_0^2}\right) C_2^k (k+3)^{2(k+3)} \|w\|^2_{L^2(Q_0)}, \tag{3.1.62}$$

where the constant C_4 in (3.1.61) and (3.1.62) depends on λ_2 and $L\rho^{-1}$ only. Finally, (3.1.55), (3.1.61) and (3.1.62) yield the convergence in $H^2(\Delta_{\frac{3}{4}\rho})$ of the power series (3.1.51) in the disk D_{ρ_1}, where ρ_1 is given by (3.1.47). □

Let us denote, for $x \in \Delta_{\frac{3}{4}\rho}$,

$$W(x,z) = \sum_0^\infty \partial_y^k w(x,0)\frac{z^k}{k!}, \quad \text{for } z \in D_{\rho_1},$$

$$v(x,\xi) = W(x, i\xi), \quad \text{for } |\xi| < \rho_1.$$

Step 2. *For every $\xi \in (-\rho_2, \rho_2)$ we have*

$$\int_{\Delta_{\rho(\xi)}} (b(x) v_\xi^2(x,\xi) + \kappa \nabla v(x,\xi) \cdot \nabla v(x,\xi)) dx \leq \int_{\Delta_{\frac{\rho}{2}}} \kappa \nabla f(x) \cdot \nabla f(x) dx, \tag{3.1.63}$$

where

$$\rho(\xi) = \frac{\rho}{2} - \sqrt{\lambda}\lambda_1 |\xi|. \tag{3.1.64}$$

Proof of Step 2. First, let us observe that v is the solution to the following Cauchy problem for a hyperbolic equation:

$$b(x) v_{\xi\xi}(x,\xi) - \operatorname{div}(\kappa(x) \nabla v(x,\xi)) = 0, \quad \text{in } \Delta_{\frac{\rho}{2}} \times (-\rho_1, \rho_1), \tag{3.1.65a}$$

$$v(x,0) = f(x), \quad \text{in } \Delta_{\frac{\rho}{2}}, \tag{3.1.65b}$$

$$v_\xi(x,0) = 0, \quad \text{in } \Delta_{\frac{\rho}{2}}. \tag{3.1.65c}$$

We shall derive estimate (3.1.63) from an energy estimate for equation (3.1.65a). To this aim, let us denote

$$E(\xi) = \frac{1}{2} \int_{\Delta_{\rho(\xi)}} (b(x) v_\xi^2(x,\xi) + \kappa(x) \nabla v(x,\xi) \cdot \nabla v(x,\xi)) dx.$$

Since $\xi \to v(\cdot, \xi)$ is an analytic function from $(-\rho_1, \rho_1)$ to $H^2(\Delta_{\frac{\rho}{2}})$, we have that $\partial_\xi^k v(\cdot, \xi) \in H^{3/2}(\partial \Delta_{\rho(\xi)})$ for every $\xi \in (-\rho_2, \rho_2)$ and for every $k \geq 1$, where ρ_2 is given by (3.1.48). Therefore, for every $\xi \in (-\rho_2, \rho_2)$, we may write the following equalities:

$$E(\xi) = \frac{1}{2}\int_0^{\rho(\xi)} d\eta \int_{\partial\Delta_\eta} (b(x)v_\xi^2(x,\xi) + \kappa(x)\nabla v(x,\xi)\cdot\nabla v(x,\xi))ds,$$

$$E'(\xi) = \int_0^{\rho(\xi)} d\eta \int_{\partial\Delta_\eta} (b(x)v_\xi(x,\xi)v_{\xi\xi}(x,\xi) + \kappa(x)\nabla v(x,\xi)\cdot\nabla v_\xi(x,\xi))ds$$

$$-\frac{\sqrt{\lambda}\lambda_1}{2}\int_{\partial\Delta_{\rho(\xi)}} (b(x)v_\xi^2(x,\xi) + \kappa(x)\nabla v(x,\xi)\cdot\nabla v(x,\xi))ds$$

$$= \int_{\partial\Delta_{\rho(\xi)}} \kappa(x)\nabla v(x,\xi)\cdot\nu v_\xi(x,\xi)ds$$

$$-\frac{\sqrt{\lambda}\lambda_1}{2}\int_{\partial\Delta_{\rho(\xi)}} (b(x)v_\xi^2(x,\xi) + \kappa(x)\nabla v(x,\xi)\cdot\nabla v(x,\xi))ds,$$

where ν denotes the outer unit normal to $\Delta_{\rho(\xi)}$. We have

$$|\kappa(x)\nabla v(x,\xi)\cdot\nu v_\xi(x,\xi)| \le (\kappa(x)\nabla v(x,\xi)\cdot\nabla v(x,\xi))^{1/2}(\kappa(x)\nu\cdot\nu)^{1/2}|v_\xi|$$

$$\le \frac{\sqrt{\lambda}}{2}(v_\xi^2(x,\xi) + \kappa(x)\nabla v(x,\xi)\cdot\nabla v(x,\xi)).$$

Therefore

$$E'(\xi) \le \frac{\sqrt{\lambda}}{2}\int_{\partial\Delta_{\rho(\xi)}} (v_\xi^2(x,\xi) + \kappa(x)\nabla v(x,\xi)\cdot\nabla v(x,\xi))ds$$

$$-\frac{\sqrt{\lambda}\lambda_1}{2}\int_{\partial\Delta_{\rho(\xi)}} (\lambda_1^{-1}v_\xi^2(x,\xi) + \kappa(x)\nabla v(x,\xi)\cdot\nabla v(x,\xi))ds$$

$$= \frac{\sqrt{\lambda}}{2}(1-\lambda_1)\int_{\partial\Delta_{\rho(\xi)}} (\kappa(x)\nabla v(x,\xi)\cdot\nabla v(x,\xi)ds \le 0.$$

Hence $E(\cdot)$ is decreasing, so that $E(\xi) \le E(0)$ and (3.1.63) follows. □

Proof of Proposition 3.1.4 (Conclusion). For every $z \in D_{\rho_1}$ let us set

$$G(z) = \int_{\Delta_{\rho_3}} (b(x)W_z^2(x,z) + \kappa(x)\nabla W(x,z)\cdot\nabla W(x,z))dx,$$

and let

$$\epsilon^2 = \int_{\Delta_{\frac{\rho}{2}}} \kappa(x)\nabla f(x)\cdot\nabla f(x)dx.$$

Let $\rho_1' \in (0,\rho_1)$. By (3.1.55) and (3.1.61) we obtain

$$|G(z)| \le \frac{C}{L\rho^2(1-\rho_1'\rho_1^{-1})^8}\|w\|^2_{L^2(Q_0)}, \quad \text{for every } z \in D_{\rho_1'}, \tag{3.1.66}$$

where C depends on λ_2 and $L\rho^{-1}$ only. On the other side (3.1.63) gives

$$|G(i\xi)| \le \epsilon^2, \quad \text{for every } \xi \in (-\rho_2,\rho_2). \tag{3.1.67}$$

From (3.1.66), (3.1.67) and the analytic continuation estimate (see [Is1]) we obtain

$$\int_{\Delta_{\rho_3}} (b(x)w_y^2(x,y) + \kappa(x)\nabla w(x,y)\cdot\nabla w(x,y))dx = G(y)$$

$$\leq \frac{1}{(1-\rho_1'\rho_1^{-1})^8}\left(\frac{C}{L\rho^2}\|w\|^2_{L^2(Q_0)} + \int_{\Delta_{\frac{\rho}{2}}} \kappa(x)\nabla f(x)\cdot\nabla f(x)dx\right)^{1-\omega(y,0)}$$

$$\times\left(\int_{\Delta_{\frac{\rho}{2}}} \kappa(x)\nabla f(x)\cdot\nabla f(x)dx\right)^{\omega(y,0)}, \tag{3.1.68}$$

where $\omega(y,\xi)$ is the harmonic measure of $\{i\xi \text{ s.t. } \xi\in[-\frac{\rho_2}{2},\frac{\rho_2}{2}]\}$ with respect to $\{y+i\xi\in\mathbb{C}\text{ s.t. } y^2+\xi^2=(\rho_1')^2\}$ and C depends on λ_2 and $L\rho^{-1}$ only. Now, let us choose $\rho_1'=\frac{3}{4}\rho_1$, so that $\frac{\rho_2}{2}<\rho_1'<\rho_1$. We have that $\omega(y,0)\geq\beta>0$ for every $y\in(-\frac{3}{8}\rho_1,\frac{3}{8}\rho_1)$, where β depends on λ and Λ only. Therefore estimate (3.1.50) follows by (3.1.68). □

Lemma 3.1.5. *Let w be the solution to the Cauchy problem*

$$\operatorname{div}(\kappa(x)\nabla w(x,y)) + b(x)w_{yy}(x,y) = 0, \quad \text{in } \Delta_R\times(-\sqrt{2}\delta,+\sqrt{2}\delta), \tag{3.1.69a}$$

$$w(x,0)=f(x), \quad \text{in } \Delta_R, \tag{3.1.69b}$$

$$w_y(x,0)=0, \quad \text{in } \Delta_R, \tag{3.1.69c}$$

where κ satisfies (3.1.1), *b satisfies* (3.1.2) *and δ is given by* (3.1.32). *For every $r\leq\theta_1 R$ we have*

$$\int_{B_r} w^2dxdy \leq Cr^{\bar\beta}\left(\int_{B_{\tilde\rho}} w^2dxdy\right)^{1-\bar\beta}\left(\int_{\Delta_{\rho/2}} f^2dx\right)^{\bar\beta}, \tag{3.1.70}$$

where θ_1, $0<\theta_1<1$, $\bar\beta$, $0<\bar\beta<1$ and $C\geq1$ depend on λ_2 only, $\rho=\frac{8}{3}\sqrt{2}e\pi\lambda_2 r$ and $\tilde\rho=2\sqrt{2}\rho$.

Proof of Lemma 3.1.5. Let us denote

$$\theta_1=\frac{3}{\sqrt{2}(8e\pi\lambda_2)^2},$$

and let r be such that $0<r\leq\theta_1 R$. Let us choose

$$\rho=L=\frac{8\sqrt{2}}{3}e\pi\lambda_2 r$$

in estimate (3.1.50). This choice gives $r=\frac{3}{8}\rho_1$, where ρ_1 is defined by (3.1.47). Let us denote

$$\tilde\rho=2\sqrt{2}\rho,$$

and let us notice that $\tilde\rho<\sqrt{2}\delta$. We have

$$B_r\subset\Delta_{\rho/4}\times(-\frac{3}{8}\rho_1,\frac{3}{8}\rho_1)\subset\Delta_{2\rho}\times(-2L,2L)\subset B_{\tilde\rho}\subset\Delta_R\times(-\sqrt{2}\delta,\sqrt{2}\delta). \tag{3.1.71}$$

Integrating both the sides of inequality (3.1.50), we obtain, by the inclusions (3.1.71),

(3.1.72)
$$\int_{B_r^+} |\nabla w|^2 dxdy \leq Cr \left(\frac{1}{r^3} \|w\|^2_{L^2(B_\rho^+)} + \|\nabla f\|^2_{L^2(\Delta_{\rho/2})} \right)^{1-\beta} \|\nabla f\|^{2\beta}_{L^2(\Delta_{\rho/2})},$$

where C only depends on λ_2. By the standard elliptic estimate (see, for instance, [GT])

$$\rho^{1+\alpha} |\nabla w|_{\alpha, B^+_{\frac{3}{4}\rho}} \leq \frac{C}{\rho^{\frac{n+1}{2}}} \|w\|_{L^2(B_\rho^+)}, \quad \text{for every } \alpha \in (0,1), \tag{3.1.73}$$

where C depends on λ_2, Λ_2 and α only, and by (A.3) we obtain

(3.1.74)
$$\rho^2 \int_{\Delta\rho/2} |\nabla f|^2 dx \leq C \left(\int_{\Delta\rho/2} f^2 dx \right)^{\frac{\alpha}{1+\alpha}} \left(\int_{\Delta\rho/2} f^2 dx + \frac{1}{\rho} \|w\|^2_{L^2(B_\rho^+)} \right)^{\frac{1}{1+\alpha}},$$

where C depends on λ_2, Λ_2 and α only. By (A.4) and by the Caccioppoli inequality we have

(3.1.75)
$$\int_{\Delta\rho/2} f^2 dx \leq c \left(\frac{1}{\rho} \int_{B^+_{\frac{3}{4}\rho}} w^2 dxdy + \rho \int_{B^+_{\frac{3}{4}\rho}} |\nabla w|^2 dxdy \right) \leq \frac{C}{\rho} \int_{B_\rho^+} w^2 dxdy,$$

where C depends on λ_2 only. By (3.1.74) and (3.1.75), we have

$$\rho^2 \int_{\Delta\rho/2} |\nabla f|^2 dx \leq C \left(\int_{\Delta\rho/2} f^2 dx \right)^{\frac{\alpha}{1+\alpha}} \left(\frac{1}{\rho} \|w\|^2_{L^2(B_\rho^+)} \right)^{\frac{1}{1+\alpha}}, \tag{3.1.76}$$

where C depends on λ_2, Λ_2 and α only. By (3.1.75) and (3.1.76), we have

$$\rho^2 \int_{\Delta\rho/2} |\nabla f|^2 dx \leq \frac{C}{\rho} \|w\|^2_{L^2(B_\rho^+)}, \tag{3.1.77}$$

where C depends on λ_2, Λ_2 and α only. By (3.1.72), (3.1.76) and (3.1.77), we obtain

$$\int_{B_r^+} |\nabla w|^2 dxdy \leq Cr \left(\frac{1}{r^2} \int_{\Delta\rho/2} f^2 dx \right)^{\bar{\beta}} \left(\frac{1}{r^3} \|w\|^2_{L^2(B_\rho^+)} \right)^{1-\bar{\beta}}, \tag{3.1.78}$$

where C depends on λ_2, Λ_2 and α only and $\bar{\beta} = \frac{\beta\alpha}{1+\alpha}$. By (A.5), (3.1.78) and (3.1.75) we have

$$\int_{B_r^+} w^2 dxdy \leq c \left(r \int_{\Delta_r} f^2 dx + r^2 \int_{B_r^+} |\nabla w|^2 dxdy \right)$$

$$\leq Cr \left(\int_{\Delta_{\rho/2}} f^2 dx \right)^{\bar{\beta}} \left(\frac{1}{r} \|w\|^2_{L^2(B_\rho^+)} \right)^{1-\bar{\beta}}, \tag{3.1.79}$$

where C depends on λ_2, Λ_2 and α only. Choosing $\alpha = \frac{1}{2}$, we obtain (3.1.70). □

Now, let us recall the three spheres inequality for elliptic equations. To this aim, let as assume that $\sigma(x)$ is a given function from the ball in $\mathbb{R}^m$ $B_{\bar{R}}$ with values $m \times m$ symmetric matrices satisfying the following conditions:

(3.1.80a)
$$\lambda^{-1}|\xi|^2 \le \sigma(x)\xi\cdot\xi \le \lambda|\xi|^2, \quad \text{for every } x \in B_{\bar{R}} \text{ and } \xi \in \mathbb{R}^m, \quad (\textit{ellipticity})$$

(3.1.80b)
$$|\sigma(x') - \sigma(x'')| \le \frac{\Lambda}{R_0}|x' - x''|, \quad \text{for every } x', x'' \in B_{\bar{R}}. \quad (\textit{Lipschitz continuity})$$

The proof of the following Lemma 3.1.6 follows from [K] with some slight changes.

Lemma 3.1.6 (Kukavica)**.** *Let $w \in H^1(B_{\bar{R}})$ be a solution to*

$$div(\sigma(x)\nabla w(x)) = 0, \quad in\ B_{\bar{R}}, \tag{3.1.81}$$

where σ satisfies (3.1.80)*. For every $0 < r_1 < r_2 < \frac{r_3}{2\lambda} \le \theta_2 \bar{R}$ we have*

$$\int_{B_{r_2}(0)} w^2(x)dx$$

(3.1.82)
$$\le C\left(\frac{r_3}{r_2}\right)^C \left(\int_{B_{r_1}(0)} w^2(x)dx\right)^{\gamma(r_1,r_2,r_3)} \left(\int_{B_{r_3}(0)} w^2(x)dx\right)^{1-\gamma(r_1,r_2,r_3)},$$

where

$$\gamma(r_1,r_2,r_3) = \frac{\log\left(\frac{1}{2} + \frac{r_3}{2\lambda r_2}\right)}{\log\left(\frac{1}{2} + \frac{r_3}{2\lambda r_2}\right) + C\log\frac{2\lambda r_2}{r_1}}, \tag{3.1.83}$$

and θ_2, $0 < \theta_2 < 1$, depends on Λ only and $C > 0$ depends on λ and Λ only.

Proof of Theorem 3.1.1. For fixed $t_0 \in (0,T)$, let us consider the function w introduced above, which satisfies (3.1.45). Let r_1, r_2, r_3 be such that

$$0 < r_1 < r_2 < (4\max\{\sqrt{2}, \lambda_2\})^{-1} r_3 < r_3 \le \bar{\theta} R,$$

with $\bar{\theta} = \min\left\{\theta_1, \frac{\theta_2 \delta}{\sqrt{2}\lambda_2 R}\right\}$, where θ_1, θ_2 have been introduced in Lemma 3.1.5 and Lemma 3.1.6 respectively and δ is given by (3.1.32). Let $r_2' = (1-\mu)r_2 + \mu r_3$, where $\mu = \frac{1}{4\lambda_2 - 1}$, and let $a = \frac{3}{4\sqrt{2}e\pi\lambda_2}$. We have that $0 < ar_1 < r_1 < r_2 < r_2' < \frac{r_3}{2\lambda_2}$. By applying Lemma 3.1.6 to the triplet of radii ar_1, r_2', r_3, we obtain

$$\int_{B_{r_2'}} w^2 dxdy \le C\left(\frac{r_3}{r_2}\right)^C \left(\int_{B_{ar_1}} w^2 dxdy\right)^{\gamma'} \left(\int_{B_{r_3}} w^2 dxdy\right)^{1-\gamma'}, \tag{3.1.84}$$

where $\gamma' = \gamma(ar_1, r_2', r_3)$, with γ defined by (3.1.83), and where C depends on λ_2 and Λ_2 only. Let $\bar{r}_2 = \frac{1}{2}(r_2 + r_2')$. By the Caccioppoli inequality, by (A.4) and by the choice of μ, it follows that

$$\int_{B_{r_2'}} w^2 dxdy = \frac{1}{2}\int_{B_{r_2'}} w^2 dxdy + \frac{1}{2}\int_{B_{r_2'}} w^2 dxdy$$
$$\ge \frac{1}{2}\int_{B_{\bar{r}_2}} w^2 dxdy + \frac{c}{\lambda_2^4}(r_2' - \bar{r}_2)^2 \int_{B_{\bar{r}_2}} |\nabla w|^2 dxdy$$

$$\geq C(r_3 - r_2)\left(\frac{\bar{r}_2}{\bar{r}_2^2 - r_2^2}\int_{B_{\bar{r}_2}} w^2 dxdy + \bar{r}_2 \int_{B_{\bar{r}_2}} |\nabla w|^2 dxdy\right)$$

$$\geq C(r_3 - r_2)\int_{\Delta_{r_2}} w^2(x,0)dx = C(r_3 - r_2)\int_{\Delta_{r_2}} u^2(x,t_0)dx, \tag{3.1.85}$$

where C depends on λ_2 only. On the other hand, by applying Lemma 3.1.5 we can estimate the integral $\int_{B_{ar_1}} w^2 dxdy$ appearing in (3.1.84), obtaining

(3.1.86)
$$\int_{B_{r_2'}} w^2 dxdy \leq C\left(\frac{r_3}{r_2}\right)^C r_1^{\bar{\beta}\gamma'}\left(\int_{\Delta_{r_1}} u^2(x,t_0)dx\right)^{\bar{\beta}\gamma'}\left(\int_{B_{r_3}} w^2 dxdy\right)^{1-\bar{\beta}\gamma'},$$

where C depends on λ_2 and Λ_2 only. Now, let us estimate $\int_{B_{r_3}} w^2 dxdy$. By the definition of w, and recalling that $r_3 \leq \frac{\delta}{2}$, we have

(3.1.87)
$$|w(x,y)| = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\hat{u}(x,\mu)| e^{\sqrt{|\mu|/2}r_3} d\mu \leq C(\delta)\left(\int_{-\infty}^{+\infty} |\hat{u}(x,\mu)|^2 e^{\delta\sqrt{|\mu|}} d\mu\right)^{1/2},$$

where

$$C(\delta) = \frac{1}{2\pi}\left(\int_{-\infty}^{+\infty} e^{\left(\frac{1}{\sqrt{2}}-1\right)\delta\sqrt{|\mu|}} d\mu\right)^{1/2} = \left(1 - \frac{1}{\sqrt{2}}\right)^{-1}\frac{1}{\pi\delta}.$$

By (3.1.31) and (3.1.87) we have

$$\int_{\Delta_{r_3}} w^2(x,y)dx \leq C^2(\delta)\int_{-\infty}^{+\infty}\int_{\Delta_{r_3}} |\hat{u}(x,\mu)|^2 e^{\delta\sqrt{|\mu|}} dxd\mu$$

$$\leq cC^2(\delta)C_1^2\lambda_2^2 H^2\left(T + \frac{1}{a_R}\right)^2\int_{-\infty}^{+\infty} e^{-\delta\sqrt{|\mu|}} d\mu. \tag{3.1.88}$$

Since $\int_{-\infty}^{+\infty} e^{-\delta\sqrt{|\mu|}} d\mu = \frac{4}{\delta^2}$, from the definition of a_R and δ and from (3.1.88) we obtain

$$\int_{B_{r_3}} w^2 dxdy \leq \int_0^{r_3}\int_{\Delta_{r_3}} w^2 dxdy \leq CC_1^2 r_3 H^2\left(\frac{T^2}{R^4} + 1\right), \tag{3.1.89}$$

where C depends on λ_2 only. From (3.1.85), (3.1.86) and (3.1.89), choosing $t_0 \in \left(0, \frac{T}{2}\right)$ in order to control the constant C_1 given by (3.1.13), we obtain (3.1.7*a*). By integrating (3.1.7a) over $\left(0, \frac{T}{2}\right)$ and using the Hölder inequality we obtain (3.1.7b). □

Theorem 3.1.1′. (Three Spheres Inequalities and Three Cylinders Inequalities in the Interior). *Let* $u \in H^{2,1}(\Delta_{2R} \times (0,T))$ *be a solution to* (3.1.5a) *and let the assumptions* (3.1.1) – (3.1.2) *be satisfied. For every* r_1, r_2, r_3 *such that* $0 < r_1 <$

$r_2 < (4\max\{\sqrt{2},\lambda_2\})^{-1}r_3 < r_3 \le \hat{\theta}_1 R$, we have

$$\int_{\Delta_{r_2}} u^2(x,t_0)dx$$

(3.1.90a)

$$\le \tilde{C}_1 \frac{r_3}{r_3-r_2}\left(\frac{r_3}{r_2}\right)^C \left(\left(1+\frac{T^2}{R^4}\right)H^2\right)^{1-\bar{\beta}\gamma'} \left(\int_{\Delta_{r_1}} u^2(x,t_0)dx\right)^{\bar{\beta}\gamma'}, \forall t_0 \in (0,T),$$

where H and γ' are defined by (3.1.6) *and* (3.1.8) *respectively, $\bar{\beta}$, $0<\bar{\beta}<1$, depends on λ_2 only, C depends on λ_2 and Λ_2 only, $\hat{\theta}_1$, $0<\hat{\theta}_1<1$, depends on λ_2, Λ_2 and $\frac{t_0}{R^2}$ only, $\hat{\theta}_1 = O(t_0)$ as $t_0 \to 0$, and $\tilde{C}_1$ depends on λ_2, Λ_2, $\frac{R_0^2}{T}$ and $\frac{T}{T-t_0}$ only. Let $\sigma \in (0,\frac{1}{2})$. For every r_1, r_2, r_3 such that $0 < r_1 < r_2 < (4\max\{\sqrt{2},\lambda_2\})^{-1}r_3 < r_3 \le \hat{\theta}_2 R$, we have*

$$\int_{\sigma T}^{(1-\sigma)T}\int_{\Delta_{r_2}} u^2(x,t)dxdt$$

(3.1.90b)

$$\le \tilde{C}_2 \frac{r_3}{r_3-r_2}\left(\frac{r_3}{r_2}\right)^C \left(T\left(1+\frac{T^2}{R^4}\right)H^2\right)^{1-\bar{\beta}\gamma'} \left(\int_{\sigma T}^{(1-\sigma)T}\int_{\Delta_{r_1}} u^2(x,t)dxdt\right)^{\bar{\beta}\gamma'},$$

where H, γ', $\bar{\beta}$ and C are as above, $\hat{\theta}_2$, $0<\hat{\theta}_2<1$, depends on λ_2, Λ_2, σ and $\frac{T}{R^2}$ only, and $\tilde{C}_2$ depends on λ_2, Λ_2, $\frac{R_0^2}{T}$ and σ only.

Proof. Let us fix $t_0 \in (0,T)$ and, let us denote by u_1, u_2 the weak solutions to problem (3.1.9) and to the following problem:

$$b(x)u_{2,t}(x,t) - \text{div}(\kappa(x)\nabla u_2(x,t)) = 0, \quad \text{in } \Delta_{2R}\times(0,+\infty), \tag{3.1.91a}$$

$$u_2 = 0, \quad \text{on } \partial\Delta_{2R}\times[0,+\infty), \tag{3.1.91b}$$

$$u_2 = u, \quad \text{on } \Delta_{2R}\times\{0\}, \tag{3.1.91c}$$

respectively. We have that $u = u_1 + u_2$ in $\Delta_{2R}\times[0,t_0]$ and, in particular,

$$u(\cdot,t_0) = u_1(\cdot,t_0) + u_2(\cdot,t_0), \quad \text{in } \Delta_{2R}.$$

By some slight changes in the proofs of Lemma 3.1.2 and Lemma 3.1.3, we have that (3.1.12) – (3.1.13) and (3.1.31) – (3.1.32) continue to hold if we replace $\tilde{u}$ by u_1 and $\hat{u}(x,\mu)$ by

$$\hat{u}_1(x,\mu) = \int_{-\infty}^{+\infty} e^{-\mu i t}u_1(x,t)dt = \int_0^{+\infty} e^{-\mu i t}u_1(x,t)dt, \quad \text{for every } \mu\in\mathbb{R}.$$

Moreover, the function $w_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it_0\mu}\hat{u}_1(x,\mu)\cosh\sqrt{-i\mu}\, y d\mu$, defined for $x\in\Delta_R$, $|y|<\sqrt{2}\delta$, with δ given by (3.1.32), is a solution to the elliptic equation (3.1.45a) and satisfies the following conditions:

$$w_1(x,0) = u_1(x,t_0), \quad \text{in } \Delta_R,$$

$$w_{1,y}(x,0) = 0, \quad \text{in } \Delta_R.$$

Now, let us define a continuation of $u_2(\cdot,t_0)$ to a solution of the elliptic equation (3.1.45a). Let μ_k, φ_k be the eigenvalues and eigenfunctions associated to problem

(3.1.20), respectively (as in the proof of Lemma 3.1.2), and let α_k be the Fourier coefficients

$$\alpha_k = \int_{\Delta_{2R}} u(x,0)b(x)\varphi_k(x)dx.$$

We have

$$u_2(x,t) = \sum_{k=1}^{\infty} \alpha_k e^{\mu_k t}\varphi_k(x), \quad \text{in } \Delta_{2R} \times (0,+\infty).$$

Let us define

$$w_2(x,y) = \sum_{k=1}^{\infty} \alpha_k e^{\mu_k t_0}\varphi_k(x) \cosh\sqrt{|\mu_k|}\, y, \quad \text{in } \Delta_{2R} \times [-t_0\sqrt{a_R}, t_0\sqrt{a_R}].$$

We have that $w_2 \in H^1_{loc}(\Delta_{2R} \times [-t_0\sqrt{a_R}, t_0\sqrt{a_R}])$ satisfies

$$\operatorname{div}(\kappa(x)\nabla w_2(x,y)) + b(x)w_{2,yy}(x,y) = 0, \quad \text{in } \Delta_{2R} \times [-t_0\sqrt{a_R}, t_0\sqrt{a_R}],$$

$$w_2(x,0) = u_2(x,t_0), \quad \text{in } \Delta_{2R},$$

$$w_{2,y}(x,0) = 0, \quad \text{in } \Delta_{2R},$$

and moreover w_2 is even with respect to the variable y.

Setting $\bar{\delta} = \min\{\sqrt{2}\delta, t_0\sqrt{a_R}\} = CR$, with C only depending on λ, λ_1, $\frac{t_0}{R^2}$, and defining

$$w = w_1 + w_2 \quad \text{in } \Delta_R \times (-\bar{\delta}, +\bar{\delta}),$$

we obtain that $w \in H^1_{loc}(\Delta_R \times (-\bar{\delta}, \bar{\delta}))$ satisfies the following elliptic equation

$$\operatorname{div}(\kappa(x)\nabla w(x,y)) + b(x)w_{yy}(x,y) = 0, \quad \text{in } \Delta_R \times (-\bar{\delta}, +\bar{\delta}),$$

and the conditions

$$w(x,0) = u(x,t_0), \quad \text{in } \Delta_R,$$

$$w_y(x,0) = 0, \quad \text{in } \Delta_R.$$

Moreover, w is even with respect to the variable y. Let us prove that

$$\int_{B_r^+} w_2^2 dxdy \le \lambda_1^2 r H^2, \tag{3.1.92}$$

for $r \le \min\{\sqrt{a_R}t_0, 2R\}$.

We have

$$\int_{B_r^+} w_2^2 dxdy \le \lambda_1 \int_0^r \left(\int_{\Delta_{2R}} w_2^2(x,y)b(x)dx \right) dy$$

$$= \lambda_1 \int_0^r \sum_{k=1}^{\infty} \alpha_k^2 e^{2\mu_k t_0} \cosh^2\sqrt{|\mu_k|}y dy \le \lambda_1 r \sum_{k=1}^{\infty} \alpha_k^2 e^{-2\sqrt{|\mu_k|}(\sqrt{|\mu_k|}t_0 - r)}$$

$$\le \lambda_1 r \sum_{k=1}^{\infty} \alpha_k^2 = \lambda_1 r \int_{\Delta_{2R}} u^2(x,0)b(x)dx \le \lambda_1^2 r H^2.$$

Now, let r_1, r_2, r_3 be such that $0 < r_1 < r_2 < (4\max\{\sqrt{2}, \lambda_2\})^{-1} r_3 < r_3 \le \hat{\theta}_1 R$, where $\hat{\theta}_1 = \min\left\{\bar{\theta}, \frac{1}{\lambda_2\sqrt{c_P}}\frac{t_0}{R^2}\right\}$, where $\bar{\theta}$ has been introduced in Theorem 3.1.1. By

(3.1.92) and by noticing that estimate (3.1.89) in the proof of Theorem 3.1.1 holds also for w_1, with the same values of the constants, we obtain

$$\int_{B_{r_3}} w^2 dxdy \leq CC_1^2 r_3 H^2 \left(\frac{T^2}{R^4}+1\right),$$

where C depends on λ_2 only and C_1 is defined by (3.1.13). From this point we can repeat the proof of Theorem 3.1.1, obtaining (3.1.90a), with the stated dependence of the constants. Now, let $\sigma \in (0, \frac{1}{2})$ and let r_1, r_2, r_3 be such that $0 < r_1 < r_2 < (4\max\{\sqrt{2}, \lambda_2\})^{-1} r_3 < r_3 \leq \hat{\theta}_2 R$, with $\hat{\theta}_2 = \min\left\{\bar{\theta}, \frac{1}{\lambda_2\sqrt{cP}}\frac{\sigma T}{R^2}\right\}$. By integrating (3.1.90a) over $(\sigma T, (1-\sigma)T)$ and using the Hölder inequality, we obtain (3.1.90b). □

Corollary 3.1.7. (Strong Unique Continuation on the Characteristic Planes $t = t_0$). *Let* $u \in H^{2,1}(\Delta_{2R} \times (0,T))$ *be a solution to* (3.1.5a) *and let* (3.1.1) – (3.1.2) *be satisfied. Let* $t_0 \in (0,T)$. *If, for every* $k \in \mathbb{N}$,

$$\int_{\Delta_r} u^2(x,t_0)dx = O(r^k), \quad \textit{as } r \to 0, \tag{3.1.93}$$

then

$$u(\cdot, t_0) \equiv 0, \quad \textit{in } \Delta_{2R}. \tag{3.1.94}$$

Proof. Let $r_3 = \hat{\theta}_1 R$, $r_2 = \frac{1}{2}(4\max\{\sqrt{2}, \lambda_2\})^{-1} r_3$. Let us fix $k \in \mathbb{N}$. By (3.1.93) we have, for r_1 small enough,

$$R^{-n}\int_{\Delta_{r_1}} u^2(x,t_0)dx \leq \left(\frac{r_1}{R}\right)^k. \tag{3.1.95}$$

By Theorem 3.1.1′ and by (3.1.95) we have

$$R^{-n}\int_{\Delta_{r_2}} u^2(x,t_0)dx \leq C'(R^{-n}H^2)^{1-\bar{\beta}\gamma'}\left(\frac{r_1}{R}\right)^{k\bar{\beta}\gamma'}, \tag{3.1.96}$$

where $\bar{\beta}$ and γ' are the same of Theorem 3.1.1′ and C' depends on on λ_2, Λ_2, $\frac{R^2}{T}$ and $\frac{T}{T-t_0}$ only. Passing to the limit in (3.1.96) as $r_1 \to 0$, we have

$$R^{-n}\int_{\Delta_{r_2}} u^2(x,t_0)dx \leq C'(R^{-n}H^2)^{1-\bar{\beta}\gamma'} e^{-\bar{C}k}, \tag{3.1.97}$$

where $\bar{C} > 0$ depends on λ_2 and Λ_2 only. Now, passing to the limit in (3.1.97) as $k \to \infty$, we obtain

$$R^{-n}\int_{\Delta_{r_2}} u^2(x,t_0)dx = 0, \tag{3.1.98}$$

so that $u(\cdot, t_0) \equiv 0$ in Δ_{r_2}. By iteration, (3.1.94) follows. □

Subsection 3.2 (Three Spheres Inequalities and Three Cylinders Inequalities at the Boundary).

Theorem 3.2.1. (Three Spheres Inequalities and Three Cylinders Inequalities at the Boundary). *Let* Ω *be a domain in* $\mathbb{R}^n$ *such that*

$$\partial\Omega \textit{ is of class } C^{1,1} \textit{ with constants } R_0, E. \tag{3.2.1}$$

Let $u \in H^{2,1}(\Omega \times (0,T))$ *be a solution to*

(3.2.2a) $$u_t(x,t) - \mathrm{div}(\kappa(x)\nabla u(x,t)) = 0, \quad \textit{in } \Omega \times (0,T],$$

satisfying

(3.2.2b) $$u(\cdot,0) = 0, \quad \textit{on } \bar{\Omega} \times \{0\},$$

(3.2.2c) $$\kappa \nabla u \cdot \nu = 0, \quad \textit{on } \Gamma \times [0,T],$$

where Γ *is an open portion of* $\partial\Omega$ *and* κ *satisfies* (3.1.1). *Let* $x_0 \in \Gamma$ *be such that*

(3.2.3) $$d(x_0, \partial\Omega \setminus \Gamma) > R_0.$$

For every r_1, r_2, $0 < r_1 < r_2 < 2r_2 \le \theta^* R_0$, *we have*

(3.2.4a)
$$\int_{\Delta_{r_2}(x_0)\cap\Omega} u^2(x,t_0)dx$$
$$\le \tilde{C}\left(\frac{R_0}{r_2}\right)^C \bar{H}^{2(1-\bar{\beta}\gamma')}\left(\int_{\Delta_{r_1}(x_0)\cap\Omega} u^2(x,t_0)dx\right)^{\bar{\beta}\gamma'}, \quad \textit{for every } t_0 \in (0,\frac{T}{2}),$$

(3.2.4b)
$$\int_0^{T/2}\int_{\Delta_{r_2}(x_0)\cap\Omega} u^2 dxdt \le \tilde{C}\left(\frac{R_0}{r_2}\right)^C \left(T\bar{H}^2\right)^{1-\bar{\beta}\gamma'}\left(\int_0^{T/2}\int_{\Delta_{r_1}(x_0)\cap\Omega} u^2 dxdt\right)^{\bar{\beta}\gamma'},$$

where

(3.2.5a) $$\gamma' = \frac{\log\left(\frac{1}{2} + C\frac{R_0}{\tilde{r}_2}\right)}{\log\left(\frac{1}{2} + C\frac{R_0}{\tilde{r}_2}\right) + C\log\frac{C\tilde{r}_2}{r_1}},$$

(3.2.5b) $$\tilde{r}_2 = \mu 2r_2 + (1-\mu)2\theta^*(4\max\{\sqrt{2}, \lambda_2'\})R_0,$$

(3.2.5c) $$\bar{H} = \sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^1(\Omega\cap\Delta_{R_0})},$$

where μ, $0 < \mu < 1$, $\bar{\beta}$, $0 < \bar{\beta} < 1$, θ^*, $0 < \theta^* < 1$, $\lambda_2' \ge 1$ *and* $C > 0$ *depend on* λ, Λ *and* E *only, and* $\tilde{C} \ge 1$ *depends on* λ, Λ, E *and* $\frac{R_0^2}{T}$ *only.*

Proof of Theorem 3.2.1. First, let us assume that $\kappa(x_0) = Id$. We fix coordinates (x', x_n) suitable for the local representation of the boundary as a graph as in Definition 2.1. Namely, we have $x_0 = 0$ and

$$\Omega \cap \Delta_{R_0}(0) = \{x \in \Delta_{R_0}(0) \text{ s.t. } x_n > \varphi(x')\},$$

where φ is a $C^{1,1}$ function on $\Delta'_{R_0}(0) \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$\|\varphi\|_{C^{1,1}(\Delta'_{R_0}(0))} \le ER_0.$$

For the reader's convenience, we recall the transformation flattening the boundary introduced in [AE], see also [AlBRV]. We can construct a map $\Phi \in C^{1,1}(\Delta_{R_2}(0), \mathbb{R}^n)$ such that

$$\Phi(\Delta_{R_2}(0)) \subset \Delta_{R_1}(0), \tag{3.2.6a}$$

$$\Phi(y', 0) = (y', \varphi(y')), \quad \text{for every } y' \in \Delta'_{R_2}(0) \subset \mathbb{R}^{n-1}, \tag{3.2.6b}$$

$$\Phi(\Delta^+_{R_2}(0)) \subset \Omega \cap \Delta_{R_1}(0), \tag{3.2.6c}$$

$$\frac{1}{2}|y - z| \leq |\Phi(y) - \Phi(z)| \leq C'_1|y - z|, \quad \text{for every } y, z \in \Delta_{R_2}(0), \tag{3.2.6d}$$

$$\frac{1}{2^n} \leq |\det D\Phi| \leq C'_2, \tag{3.2.6e}$$

$$|\det D\Phi(y) - \det D\Phi(z)| \leq \frac{C'_3}{R_0}|y - z|, \quad \text{for every } y, z \in \Delta_{R_2}(0), \tag{3.2.6f}$$

where $R_i = \bar{\theta}_i R_0$, $0 < \bar{\theta}_i < 1$, $i = 1, 2$, and C'_1, C'_2, C'_3, $\bar{\theta}_1$, $\bar{\theta}_2$ only depend on λ, Λ and E. Denoting

$$\bar{\kappa}(y) = |\det D\Phi(y)|(D\Phi^{-1})(\Phi(y))\kappa(\Phi(y))(D\Phi^{-1})^T(\Phi(y)),$$

$$v(y, t) = u(\Phi(y), t),$$

we have

$$\bar{\kappa}(0) = Id, \tag{3.2.7a}$$

$$\bar{\kappa}_{nk}(y', 0) = 0, \quad \text{for } k = 1, ..., n - 1. \tag{3.2.7b}$$

Moreover, we have that the ellipticity and Lipschitz constants λ', Λ' of $\bar{\kappa}$ in $\Delta^+_{R_2}(0)$ depend on λ, Λ and E only. For every $y \in \Delta_{R_2}(0)$, let us denote by $\kappa'(y)$ the symmetric matrix whose entries are given by

$$\kappa'_{ij}(y', y_n) = \bar{\kappa}_{ij}(y', |y_n|), \quad \text{if either } 1 \leq i, j \leq n - 1, \text{or } i = j = n,$$

$$\kappa'_{nj}(y', y_n) = \kappa'_{jn}(y', y_n) = \operatorname{sgn}(y_n)\bar{\kappa}_{jn}(y', |y_n|), \quad \text{if } 1 \leq j \leq n - 1.$$

We have that κ' satisfies the same ellipticity and Lipschitz continuity conditions as $\bar{\kappa}$.

Denoting

$$U(y, t) = v(y', |y_n|, t), \quad \text{for } y \in \Delta_{R_2}(0), t \in (0, T),$$

$$b(y) = |\det D\Phi(y', |y_n|)| \quad \text{for } y \in \Delta_{R_2}(0),$$

we have that $U \in H^{2,1}(\Delta_{R_2}(0) \times (0, T))$ is a solution to

$$b(y)U_t(y, t) - \operatorname{div}(\kappa'(y)\nabla U(y, t)) = 0, \quad \text{in } \Delta_{R_2}(0) \times (0, T), \tag{3.2.8a}$$

and satisfies

$$U(\cdot, 0) = 0, \quad \text{on } \Delta_{R_2}(0). \tag{3.2.8b}$$

Moreover, from (3.2.6d) we have that

$$\Omega \cap \Delta_{r/2}(0) \subset \Phi(\Delta^+_r(0)) \subset \Omega \cap \Delta_{C'_1 r}(0), \quad \text{for every } r \leq R_2. \tag{3.2.9a}$$

(3.2.9b) $$\Delta^{+}_{r/C'_1}(0) \subset \Phi^{-1}(\Omega \cap \Delta_r(0)) \subset \Delta^{+}_{2r}(0), \quad \text{for every } r \le R_1.$$

Let us set $\theta^* = \frac{(4\max\{\sqrt{2},\lambda'_2\})^{-1}\bar{\theta}\bar{\theta}_2}{4}$, where $\bar{\theta}$ has been introduced in Theorem 3.1.1 and $\lambda'_2 = \max\{\lambda', C'_2, 2^n\}$. Let r_1, r_2 be such that $0 < r_1 < r_2 < 2r_2 \le \theta^* R_0$. By the choice made of θ^*, we can apply (3.1.7a)-(3.1.7b) to U for the triplet of radii $s_1 = \frac{r_1}{C'_1}$, $s_2 = 2r_2$, $s_3 = \frac{\bar{\theta}R_2}{2} = 2\theta^*(4\max\{\sqrt{2},\lambda'_2\})R_0$. By simple changes of variables in the integrals we obtain $(3.2.4) - (3.2.5)$. In the general case $\kappa(x_0) \ne Id$, we can consider a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that, setting $\tilde{\kappa}(Sx) = \frac{S\kappa(x)S^T}{|\det S|}$, we have $\tilde{\kappa}(x_0) = Id$ (here, as above, we identify $x_0 = 0$). We have that, under such a transformation, the modified coefficient $\tilde{\kappa}$, the transformed domain $S(\Omega)$ and the transformed boundary portion $S(\Gamma)$ satisfy assumptions analogous to (3.1.1), (3.2.1) and (3.2.3) with constants which are dominated by the a priori constants λ, Λ, R_0, E, up to multiplicative factors which only depend on λ. Moreover, $\tilde{H} = \sup_{t\in[0,T]} \|(u\cdot S^{-1})(\cdot,t)\|_{H^1(S(\Omega\cap\Delta_{R_0}))}$ is dominated by $\bar{H}$ up to a multiplicative factor which only depends on λ. We also have that the ellipsoids $S(\Delta_r(x_0))$ satisfy

$$\Delta_{\frac{r}{\sqrt{\lambda}}}(x_0) \subset S(\Delta_r(x_0)) \subset \Delta_{\sqrt{\lambda}r}(x_0), \quad \text{for every } r > 0.$$

Therefore, by a change of variables, using the result just proved when $\kappa(x_0) = Id$, the thesis follows. □

Theorem 3.2.1′. (Three Spheres Inequalities and Three Cylinders Inequalities at the Boundary). *Let the hypotheses of Theorem 3.2.1, except* (3.2.2b)*, be satisfied. For every r_1, r_2 such that $0 < r_1 < r_2 < 2r_2 \le \theta^* R_0$, we have*

(3.2.10a)
$$\int_{\Delta_{r_2}(x_0)\cap\Omega} u^2(x,t_0)dx$$
$$\le \tilde{C}_1 \left(\frac{R_0}{r_2}\right)^C \bar{H}^{2(1-\bar{\beta}\gamma')} \left(\int_{\Delta_{r_1}(x_0)\cap\Omega} u^2(x,t_0)dx\right)^{\bar{\beta}\gamma'}, \quad \text{for every } t_0 \in (0,T),$$

where γ' is given by (3.2.5a)-(3.2.5b)*, $\bar{H}$ is given by* (3.2.5c)*, $\bar{\beta}$, $0 < \bar{\beta} < 1$, $\lambda'_2 \ge 1$ and $C > 0$ depend on λ, Λ and E only, θ^*, $0 < \theta^* < 1$, depends on λ, Λ, E and $\frac{t_0}{R_0^2}$ only, $\theta^* = O(t_0)$ as $t_0 \to 0$, and $\tilde{C}_1$ depends on λ, Λ, E, $\frac{R_0^2}{T}$ and $\frac{T}{T-t_0}$ only. Let $\sigma \in (0, \frac{1}{2})$. For every r_1, r_2 such that $0 < r_1 < r_2 < 2r_2 \le \tilde{\theta} R_0$, we have*

$$\int_{\sigma T}^{(1-\sigma)T} \int_{\Delta_{r_2}(x_0)\cap\Omega} u^2(x,t)dxdt$$

(3.2.10b) $$\le \tilde{C}_2 \left(\frac{R_0}{r_2}\right)^C (T\bar{H}^2)^{1-\bar{\beta}\gamma'} \left(\int_{\sigma T}^{(1-\sigma)T} \int_{\Delta_{r_1}(x_0)\cap\Omega} u^2(x,t)dxdt\right)^{\bar{\beta}\gamma'},$$

where $\bar{H}$, γ', $\bar{\beta}$ and C are as above, $\tilde{\theta}$, $0 < \tilde{\theta} < 1$, and $\tilde{C}_2$ depend on λ, Λ, E, σ and $\frac{R_0^2}{T}$ only.

Proof. The proof follows by slight changes of the proof of Theorem 3.2.1, with

$$\theta^* = (4\max\{\sqrt{2},\lambda'_2\})^{-1} \min\left\{\frac{t_0}{\lambda'_2\sqrt{c_P}\bar{\theta}_2 R_0^2}, \frac{\bar{\theta}\bar{\theta}_2}{4}\right\},$$

$$\tilde{\theta} = (4\max\{\sqrt{2}, \lambda_2'\})^{-1} \min\left\{\frac{\sigma T}{\lambda_2'\sqrt{c_P}\bar{\theta}_2 R_0^2}, \frac{\bar{\theta}\bar{\theta}_2}{4}\right\},$$

in the notation of the proof of Theorem 3.2.1. □

Subsection 3.3 (Stability Estimates of Continuation from Cauchy Data on Time-like Surfaces).

Theorem 3.3.1. (Stability Estimates of Continuation from Cauchy Data on Time-like Surfaces). *Let* Ω *be a domain satisfying* (3.2.1). *Let* Σ *be an open portion of* $\partial\Omega$ *satisfying*

$$\partial\Omega \cap \Delta_{R_0}(P_1) \subset \Sigma, \tag{3.3.1}$$

for some $P_1 \in \Sigma$. *Let* $u \in H^{2,1}(\Omega \times (0,T))$ *be a solution to* (3.2.2a), (3.2.2b) *satisfying*

$$\|u\|_{H^{3/2,3/4}(\Sigma\times(0,T))} \le T^{\frac{1}{2}} R_0^{\frac{n-1}{2}} \bar{\epsilon}, \tag{3.3.2}$$

$$R_0 \left\|\frac{\partial u}{\partial \nu}\right\|_{H^{1/2,1/4}(\Sigma\times(0,T))} \le T^{\frac{1}{2}} R_0^{\frac{n-1}{2}} \bar{\epsilon}. \tag{3.3.3}$$

For every $t_0 \in [0, T/2]$ *we have*

$$\|u(\cdot,t_0)\|_{L^2(\Delta_{\tilde{\theta}R_0}(P_2))} \le C R_0^{\frac{n}{2}} \left(T^{-\frac{1}{2}} R_0^{-\frac{n}{2}} \|u\|_{H^{2,1}(\Omega\times(0,T))}\right)^{1-\bar{\gamma}} \bar{\epsilon}^{\bar{\gamma}}, \tag{3.3.4}$$

where $P_2 = P_1 - \tilde{\theta} R_0 \nu$, ν *denotes the outer unit normal to* Ω *at* P_1, $\bar{\gamma}$, $0 < \bar{\gamma} < 1$, *depends on* λ *and* Λ *only,* $\tilde{\theta}$, $0 < \tilde{\theta} < \frac{1}{2}$, *depends on* λ *and* E *only,* $C \ge 1$ *depends on* λ, Λ, E *and* $\frac{R_0^2}{T}$ *only.*

Proof of Theorem 3.3.1. We may assume that $R_0^{n/2} T^{1/2} \bar{\epsilon} \le \|u\|_{H^{2,1}(\Omega\times(0,T))}$, since, otherwise, (3.3.4) is trivial. Let us denote

$$G = \Omega \cap \Delta_{R_0}(P_1),$$

$$\Sigma_0 = \partial\Omega \cap \Delta_{R_0}(P_1),$$

$$Q = \Delta_{R_0}(P_1) \times (0,T),$$

$$Q^+ = G \times (0,T),$$

$$Q^- = (\Delta_{R_0}(P_1) \setminus \bar{G}) \times (0,T).$$

Up to a rigid motion, we have

$$P_1 = 0,$$

$$G = \{x = (x', x_n) \in \Delta_{R_0} \text{ s.t. } x_n > \varphi(x')\},$$

$$\Sigma_0 = \{x = (x', x_n) \in \Delta_{R_0} \text{ s.t. } x_n = \varphi(x')\},$$

where φ is a $C^{1,1}$ function on $\Delta_{R_0} \subset \mathbb{R}^{n-1}$ satisfying

$$\varphi(0) = |\nabla\varphi(0)| = 0$$

and

$$\|\varphi\|_{C^{1,1}(\Delta_{R_0})} \le E R_0.$$

By extension theorems in Sobolev spaces [LioM], there exists $v \in H^{2,1}(Q^+)$ such that

$$v = u, \quad \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu}, \qquad \text{on } \Sigma_0, \tag{3.3.5a}$$

$$\|v\|_{H^{2,1}(Q^+)} \leq C R_0^{\frac{n}{2}} T^{\frac{1}{2}} \epsilon, \tag{3.3.5b}$$

with C only depending on E and $\frac{R_0^2}{T}$. The function $w = u - v \in H^{2,1}(Q^+)$ satisfies

$$Lw = -Lv, \quad \text{in } Q^+,$$

$$w = 0, \quad \frac{\partial w}{\partial \nu} = 0, \quad \text{on } \Sigma_0 \times (0,T),$$

where L denotes the operator

$$L\cdot = \frac{\partial \cdot}{\partial t} - \text{div}(\kappa \nabla \cdot).$$

Let us define

$$\bar{f} = \begin{cases} Lv & \text{in } Q^+, \\ 0 & \text{in } Q^-, \end{cases}$$

$$\bar{w} = \begin{cases} w & \text{in } Q^+, \\ 0 & \text{in } Q^-. \end{cases}$$

Since $w = \frac{\partial w}{\partial \nu} = 0$ on $\Sigma_0 \times (0,T)$, we have that $\bar{w} \in H^{2,1}(Q)$ and $L\bar{w} = -\bar{f}$ in Q. Let $\bar{z} \in H^{2,1}(Q)$ be the solution to

$$L\bar{z} = -\bar{f},$$

$$\bar{z}|_{\partial_P Q} = 0,$$

where $\partial_P Q$ denotes the parabolic boundary of Q. We have that $L(\bar{w} - \bar{z}) = 0$ in Q. By the regularity of φ, G satisfies an interior and exterior sphere condition at $P_1 = 0$. More precisely, setting $R_1 = \min\{\frac{1}{2}, \frac{1}{E}\} R_0$, $P_r = r e_n$, $Q_r = -r e_n$, we have that

$$\Delta_r(P_r) \subset G, \quad \Delta_r(Q_r) \subset \Delta_{R_0} \setminus \bar{G}, \quad \text{for every } r \leq R_1.$$

Since $L(\bar{w} - \bar{z}) = 0$ in $\Delta_{R_0 - R_1}(Q_r) \times (0,T)$, for every $r \leq R_0 - R_1$, we can apply (3.1.7a), making the following positions: $r_1 = r$, $r_2 = 3r$, $r_3 = 13 \max\{\sqrt{2}, \lambda_2\} r$. For every $r \leq \tilde{\theta} R_0$ and for every $t_0 \in (0, T/2)$ we have

$$\int_{\Delta_{3r}(Q_r)} (\bar{w} - \bar{z})^2(x,t_0)dx \leq C H^{2(1-\bar{\gamma})} \left(\int_{\Delta_r(Q_r)} (\bar{w} - \bar{z})^2(x,t_0)dx \right)^{\bar{\gamma}}, \tag{3.3.6}$$

where $H = \sup_{t \in [0,T]} \|(\bar{w} - \bar{z})(\cdot,t)\|_{H^1(\Delta_{R_0})}$, $\tilde{\theta} = \frac{\bar{\theta} \max\{1/2, 1-(1/E)\}}{26 \max\{\sqrt{2}, \lambda_2\}}$ depends on λ, Λ and E only, with $\bar{\theta}$ introduced in Theorem 3.1.1, $\bar{\gamma}$, $0 < \bar{\gamma} < 1$, depends on λ only, $C \geq 1$ depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. Since $\Delta_r(P_r) \subset \Delta_{3r}(Q_r)$, choosing $r = \tilde{\theta} R_0$ and recalling the definition of w, we have

$$\|(w - \bar{z})(\cdot,t_0)\|^2_{L^2(\Delta_{\tilde{\theta} R_0}(P_{\tilde{\theta} R_0}))} \leq C H^{2(1-\bar{\gamma})} \|\bar{z}(\cdot,t_0)\|^{2\bar{\gamma}}_{L^2(\Delta_{\tilde{\theta} R_0}(Q_{\tilde{\theta} R_0}))}, \tag{3.3.7}$$

where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. We have

$$\|\bar{f}\|_{L^2(Q)} = \|Lv\|_{L^2(Q^+)} \leq C\|v\|_{H^{2,1}(Q^+)}, \tag{3.3.8}$$

$$\|\bar{z}\|_{H^{2,1}(Q)} \leq \|\bar{f}\|_{L^2(Q)}, \tag{3.3.9}$$

where C depends on λ only. From (3.3.5b), (3.3.8), (3.3.9) and the triangle inequality we have

$$\|\bar{w} - \bar{z}\|_{H^{2,1}(Q)} \leq C\|u\|_{H^{2,1}(\Omega\times(0,T))}, \tag{3.3.10}$$

where C depends on E, λ and $\frac{R_0^2}{T}$ only. By using trace inequalities, we have

$$\|(\bar{w} - \bar{z})(\cdot, t_0)\|_{H^1(\Delta_{R_0})} \leq CT^{-\frac{1}{2}}\|\bar{w} - \bar{z}\|_{H^{2,1}(Q)}, \tag{3.3.11}$$

where C depends on E and $\frac{R_0^2}{T}$ only, so that, by (3.3.10), we obtain

$$H \leq CT^{-\frac{1}{2}}\|u\|_{H^{2,1}(\Omega\times(0,T))}, \tag{3.3.12}$$

with C depending on λ, E and $\frac{R_0^2}{T}$ only. From (3.3.5b), (3.3.7) – (3.3.12) we obtain

$$\|(w - \bar{z})(\cdot, t_0)\|_{L^2(\Delta_{\tilde{\theta}R_0}(P_{\tilde{\theta}R_0})} \leq C\left(R_0^{\frac{n}{2}}\bar{\epsilon}\right)^{\bar{\gamma}}\left(T^{-\frac{1}{2}}\|u\|_{H^{2,1}(\Omega\times(0,T))}\right)^{1-\bar{\gamma}}, \tag{3.3.13}$$

where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. From (3.3.5b), (3.3.8), (3.3.9), (3.3.13), the triangle inequality and trace inequalities, we have, for every $t_0 \in [0, T/2]$,

$$\|u(\cdot, t_0)\|_{L^2(\Delta_{\tilde{\theta}R_0}(P_{\tilde{\theta}R_0})} \leq C\left(R_0^{\frac{n}{2}}\bar{\epsilon}\right)^{\bar{\gamma}}\left(T^{-\frac{1}{2}}\|u\|_{H^{2,1}(\Omega\times(0,T))}\right)^{1-\bar{\gamma}}, \tag{3.3.14}$$

where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. Now (3.3.4) follows. □

4. The Inverse Problem: the Main Result

In this section Ω will be a bounded domain in $\mathbb{R}^n$, a part of which, say I (perhaps some interior connected component of $\partial\Omega$ or some inaccessible portion of the exterior component of $\partial\Omega$), is not known. Let $A = \partial\Omega \setminus I$ be the accessible part of the boundary. Given a nontrivial function g on $A \times (0, T)$, let us consider the parabolic boundary value problem

$$u_t(x, t) - \operatorname{div}(\kappa(x)\nabla u(x, t)) = 0, \quad \text{in } \Omega \times (0, T], \tag{4.1a}$$

$$u = 0 \quad \text{on } \bar{\Omega} \times \{0\}, \tag{4.1b}$$

$$\kappa\nabla u \cdot \nu = 0, \quad \text{on } I \times [0, T], \tag{4.1c}$$

$$\kappa\nabla u \cdot \nu = g, \quad \text{on } A \times [0, T], \tag{4.1d}$$

where ν denotes the exterior unit normal to Ω and κ is a function from $\mathbb{R}^n$ with values $n \times n$ symmetric matrices satisfying Lipschitz and uniform ellipticity conditions (see (4.8) below).

Given an open subset Σ of the boundary of Ω which is contained in A, we consider the inverse problem of determining I from the knowledge of $\kappa\nabla u \cdot \nu$ on $\Sigma \times [0, T]$.

i) *A priori information on the domain.*

Given R_0, $M > 0$, we assume

$$|\Omega| \leq MR_0^n. \tag{4.2}$$

Here, and in the sequel, $|\Omega|$ denotes the Lebesgue measure of Ω. We shall distinguish two nonempty parts, A, I in $\partial\Omega$, and we assume

$$I \cup A = \partial\Omega, \qquad \overset{\circ}{I} \cap \overset{\circ}{A} = \emptyset, \qquad I \cap A = \partial A = \partial I. \tag{4.3}$$

Here, interiors and boundaries are intended in the relative topology in $\partial\Omega$. Moreover we assume that we can select a portion Σ within A satisfying, for some $P_1 \in \Sigma$,

$$\partial\Omega \cap \Delta_{R_0}(P_1) \subset \Sigma, \tag{4.4}$$

and also, denoting by I^{R_0} the portion of $\partial\Omega$ of all $x \in \partial\Omega$ such that $\mathrm{dist}(x, I) < R_0$,

$$\Sigma \cap I^{R_0} = \emptyset. \tag{4.5}$$

Regarding the regularity of $\partial\Omega$, we assume that, given $E > 0$,

$$\partial\Omega \text{ is of class } C^{1,1} \text{ with constants } R_0, E. \tag{4.6}$$

Remark 4.1. Observe that (4.6) automatically implies a lower bound on the diameter of every connected component of $\partial\Omega$. Moreover, by combining (4.2) with (4.6), an upper bound on the diameter of Ω can also be obtained. Note also that (4.2), (4.6) implicitly comprise an a priori upper bound on the number of connected components of $\partial\Omega$.

ii) *Assumptions about the boundary data.*
Let us set

$$A_{R_0} = \{x \in A \text{ s.t. } \mathrm{dist}(x, I) \geq R_0\},$$

(that is: $A_{R_0} = \partial\Omega \setminus I^{R_0}$). Denoting again by g the extension by 0 of the Neumann data g appearing in problem (4.1) to $S_T = \partial\Omega \times [0, T]$, we shall assume

$$g \in H^{1/2,1/4}(S_T), \quad g \not\equiv 0, \tag{4.7a}$$

$$\mathrm{supp} g \subset A_{R_0} \times [0, T], \tag{4.7b}$$

and, for a given constant $F > 0$,

$$\frac{\|g\|_{H^{1/2,1/4}(S_T)}}{\|g\|_{L^2(S_T)}} \leq F. \tag{4.7c}$$

iii) *Assumptions about the thermal conductivity* κ.
The thermal conductivity κ is assumed to be a given function from $\mathbb{R}^n$ with values $n \times n$ symmetric matrices satisfying the following conditions for given constants λ, Λ, $\lambda \geq 1$, $\Lambda \geq 0$,

$$\lambda^{-1}|\xi|^2 \leq \kappa(x)\xi \cdot \xi \leq \lambda|\xi|^2, \quad \text{for every } x, \xi \in \mathbb{R}^n \quad (\textit{ellipticity}), \tag{4.8a}$$

$$|\kappa(x) - \kappa(y)| \leq \Lambda \frac{|x - y|}{R_0}, \quad \text{for every } x, y \in \mathbb{R}^n \quad (\textit{Lipschitz continuity}). \tag{4.8b}$$

In the sequel, we shall refer to the set of constants λ, Λ, E, M, $\frac{R_0^2}{T}$, F, as the *a priori data*.

Theorem 4.1. *Let Ω_1, Ω_2 be two domains satisfying* (4.2), (4.6)*. Let A_i, I_i, $i = 1, 2$, be the corresponding accessible and inaccessible parts of their boundaries. Let us assume that $A_1 = A_2 = A$, Ω_1, Ω_2 lie on the same side of A, and that* (4.3)−(4.5) *are satisfied by both pairs A_i, I_i. Let $u_i \in H^{2,1}(\Omega_i \times (0,T))$ be the solution to* (4.1) *when $\Omega = \Omega_i$, $i = 1,2$, and let* (4.7) *and* (4.8) *be satisfied. If, given $\epsilon > 0$, we have*

$$\|u_1 - u_2\|_{L^2(\Sigma\times(0,T))} \le T^{\frac{1}{2}} R_0^{\frac{n-1}{2}} \epsilon, \tag{4.9}$$

then we have

$$d_{\mathcal{H}}(\overline{\Omega}_1, \overline{\Omega}_2) \le R_0 \omega \left(\frac{T^{\frac{1}{2}} R_0^{\frac{n-3}{2}} \epsilon}{\|g\|_{H^{1/2,1/4}(S_T)}} \right), \tag{4.10}$$

where ω is an increasing continuous function on $[0,\infty)$ which satisfies

$$\omega(t) \le C|\log t|^{-B}, \quad \text{for every } t < 1, \tag{4.11}$$

and C, B are positive constants only depending on the a priori data.

Here $d_{\mathcal{H}}$ denotes the Hausdorff distance between bounded closed sets of $\mathbb{R}^n$.

5. Proof of Theorem 4.1

Here and in the sequel we shall denote by G the connected component of $\Omega_1 \cap \Omega_2$ such that $\Sigma \subset \bar{G}$.

The proof of Theorem 4.1 is obtained from the following sequence of propositions.

Proposition 5.1. (Three Cylinders Inequalities in the Interior). *Let Ω be a domain satisfying* (4.2)*, such that $\partial\Omega$ is of class $C^{1,1}$ with constants R_0, E. Let $u \in H^{2,1}(\Omega \times (0,T))$ be the solution to* (4.1)*, where g satisfies* (4.7a) *and κ satisfies* (4.8)*. For every $\rho > 0$ and every $x_0 \in \Omega_\rho$, we have*

$$\int_0^{T/2} \int_{\Delta_{r_2}(x_0)} u^2(x,t)dxdt \le \tilde{C} \frac{\rho}{\rho - r_2} \left(\frac{\rho}{r_2}\right)^C$$

$$\times \left(R_0^3 \left(1 + \frac{T^2}{\rho^4}\right) \|g\|^2_{H^{\frac{1}{2},\frac{1}{4}}(S_T)} \right)^{1-\bar{\beta}\gamma'} \left(\int_0^{T/2} \int_{\Delta_{r_1}(x_0)} u^2(x,t)dxdt \right)^{\bar{\beta}\gamma'}, \tag{5.1}$$

for every r_1, r_2 such that $0 < r_1 < r_2 < \theta\rho$, where

$$\gamma' = \frac{\log\left(\frac{1}{2} + \frac{(4\max\{\sqrt{2},\lambda\})\theta\rho}{2\lambda r_2'}\right)}{\log\left(\frac{1}{2} + \frac{(4\max\{\sqrt{2},\lambda\})\theta\rho}{2\lambda r_2'}\right) + C \log\frac{2\lambda r_2'}{a r_1}}, \tag{5.2a}$$

$$r_2' = \frac{4\lambda - 2}{4\lambda - 1} r_2 + \frac{1}{4\lambda - 1}(4\max\{\sqrt{2},\lambda\})\theta\rho, \tag{5.2b}$$

$$a = \frac{3}{4\sqrt{2}e\pi\lambda}, \tag{5.2c}$$

where $\bar{\beta}$, $0 < \bar{\beta} < 1$, depends on λ only, θ, $0 < \theta < 1$, and $C > 0$ depend on λ and Λ only, and $\tilde{C} \ge 1$ depends on λ, Λ, E and $\frac{R_0^2}{T}$ only.

Proposition 5.2. (Three Cylinders Inequalities at the Boundary). *Let Ω be a domain satisfying* (4.2) *and* (4.6)*. Let us assume that the accessible and inaccessible parts A, I of its boundary satisfy* (4.3) $-$ (4.5)*. Let $u \in H^{2,1}(\Omega \times (0,T))$ be the* *solution to* (4.1) *and let* (4.7) *and* (4.8) *be satisfied. Let* $x_0 \in I$. *For every* r_1, r_2, $0 < r_1 < r_2 < 2r_2 \leq \theta^* R_0$, *we have*

$$\int_0^{T/2} \int_{\Delta_{r_2}(x_0)\cap\Omega} u^2(x,t)dxdt$$

$$\leq \tilde{C}\left(\frac{R_0}{r_2}\right)^C \left(R_0^3\|g\|^2_{H^{\frac{1}{2},\frac{1}{4}}(S_T)}\right)^{1-\bar{\beta}\gamma'} \left(\int_0^{T/2}\int_{\Delta_{r_1}(x_0)\cap\Omega} u^2(x,t)dxdt\right)^{\bar{\beta}\gamma'}, \tag{5.3}$$

where

$$\gamma' = \frac{\log\left(\frac{1}{2} + C\frac{R_0}{\tilde{r}_2}\right)}{\log\left(\frac{1}{2} + C\frac{R_0}{\tilde{r}_2}\right) + C\log\frac{C\tilde{r}_2}{r_1}}, \tag{5.4a}$$

$$\tilde{r}_2 = \mu 2r_2 + (1-\mu)2\theta^*(4\max\{\sqrt{2},\lambda_2\})R_0, \tag{5.4b}$$

where μ, $0 < \mu < 1$, $\bar{\beta}$, $0 < \bar{\beta} < 1$, θ^*, $0 < \theta^* < 1$, *and* $C > 0$ *depend on* λ, Λ *and* E *only, and* $\tilde{C} \geq 1$ *depends on* λ, Λ, E *and* $\frac{R_0^2}{T}$ *only.*

Proposition 5.3. (Stability Estimate of Continuation from Cauchy Data). *Let the hypotheses of Theorem 4.1 be satisfied. We have*

$$\int_0^{T/2}\int_{\Omega_i\setminus G} u_i^2(x,t)dxdt \leq R_0^3\|g\|^2_{H^{1/2,1/4}(S_T)}\,\omega\left(\frac{R_0^{\frac{n-3}{2}}T^{\frac{1}{2}}\epsilon}{\|g\|_{H^{1/2,1/4}(S_T)}}\right), \quad i = 1,2, \tag{5.5}$$

where ω *is an increasing continuous function on* $[0,\infty)$ *which satisfies*

$$\omega(t) \leq C(\log|\log t|)^{-1/n}, \qquad \textit{for every } t < e^{-1}, \tag{5.6}$$

and $C > 0$ *depends on* λ, Λ, E, M *and* $\frac{R_0^2}{T}$ *only.*

Proposition 5.4. (Improved Stability Estimate of Continuation from Cauchy Data). *Let the hypotheses of Proposition 5.3 hold and, in addition, let us assume that there exist* $L > 0$ *and* r_0, $0 < r_0 \leq R_0$, *such that* ∂G *is of Lipschitz class with constants* r_0, L. *Then* (5.5) *holds with* ω *given by*

$$\omega(t) \leq C|\log t|^{-B}, \qquad \textit{for every } t < 1, \tag{5.7}$$

where $B > 0$ *and* $C > 0$ *depend only on* λ, Λ, E, M, $\frac{R_0^2}{T}$, L *and* R_0/r_0.

Proposition 5.5. (Stability Estimate of Continuation from the Interior). *Let* Ω *be a domain in* $\mathbb{R}^n$ *satisfying* (4.2), *such that* $\partial\Omega$ *is of class* $C^{1,1}$ *with constants* R_0, E. *Let* $u \in H^{2,1}(\Omega\times(0,T))$ *be the solution to* (4.1), *where* g *satisfies* (4.7a), (4.7c) *and* κ *satisfies* (4.8). *For every* $\rho > 0$ *and every* $x_0 \in \Omega_\rho$, *we have*

$$\int_0^{T/2}\int_{B_\rho(x_0)} u^2 dxdt \geq CR_0^3\|g\|^2_{H^{1/2,1/4}(S_T)}, \tag{5.8}$$

where $C > 0$ *depends on* λ, Λ, E, M, $\frac{R_0^2}{T}$, F *and* R_0/ρ *only.*

At this stage, we recall the notion of *modified distance* introduced in [AlBRV].

Definition 5.1. The *modified distance* between Ω_1 and Ω_2 is the number

$$d_m(\Omega_1,\Omega_2) = \max\{\sup_{x\in\partial\Omega_1} \mathrm{dist}(x,\Omega_2), \sup_{x\in\partial\Omega_2} \mathrm{dist}(x,\Omega_1)\}. \tag{5.9}$$

Notice that we obviously have

$$d_m(\Omega_1,\Omega_2) \leq d_{\mathcal{H}}(\bar{\Omega}_1,\bar{\Omega}_2), \tag{5.10}$$

but, in general, d_m does not dominate the Hausdorff distance, and indeed it does not satisfy the axioms of a distance function. This is made clear by the following example: $\Omega_1 = \Delta_1(0)$, $\Omega_2 = \Delta_1(0)\setminus\overline{\Delta_{1/2}(0)}$. In this case $d_m(\Omega_1,\Omega_2)=0$, whereas $d_{\mathcal{H}}(\Omega_1,\Omega_2)=1/2$.

The proof of the following proposition is given in [AlBRV].

Proposition 5.6 (Geometric Lemma)**.** *Let Ω_1, Ω_2 be bounded domains satisfying (4.6). There exist numbers d_0, r_0, $d_0>0$, $0<r_0\leq R_0$, for which the ratios $\frac{d_0}{R_0}$, $\frac{r_0}{R_0}$ only depend on E, such that if we have*

$$d_{\mathcal{H}}(\bar{\Omega}_1,\bar{\Omega}_2) \leq d_0, \tag{5.11}$$

then the following facts hold:

i) *There exists an absolute constant $C>0$ such that*

$$d_{\mathcal{H}}(\bar{\Omega}_1,\bar{\Omega}_2) \leq C d_m(\Omega_1,\Omega_2). \tag{5.12}$$

ii) *Any connected component G of $\Omega_1\cap\Omega_2$ has boundary of Lipschitz class with constants r_0, L, where r_0 is as above and $L>0$ only depends on E.*

Proof of Theorem 4.1. Let us denote, for brevity, $d = d_{\mathcal{H}}(\bar{\Omega}_1,\bar{\Omega}_2)$, $d_m = d_m(\Omega_1,\Omega_2)$ the modified distance defined by (5.9).

Let us prove that if $\eta>0$ is such that

$$\int_0^{T/2}\int_{\Omega_i\setminus G} u_i^2(x,t)dxdt \leq \eta, \quad i=1,2, \tag{5.13}$$

then

$$d_m \leq CR_0\left(\frac{\eta}{R_0^3\|g\|^2_{H^{1/2,1/4}(S_T)}}\right)^K, \tag{5.14}$$

$$d \leq CR_0\left(\frac{\eta}{R_0^3\|g\|^2_{H^{1/2,1/4}(S_T)}}\right)^K, \tag{5.15}$$

where $C>0$ and $K>0$ depend on λ, Λ, E, M, $\frac{R_0^2}{T}$ and F only. First, let us prove (5.14). We may assume, with no loss of generality, that there exists $x_0\in I_1\subset\partial\Omega_1$ such that $\mathrm{dist}(x_0,\Omega_2)=d_m$. By (5.13) we have

$$\int_0^{T/2}\int_{\Omega_1\cap\Delta_{d_m}(x_0)} u_1^2(x,t)dxdt \leq \eta. \tag{5.16}$$

Two cases may occur:

I) $d_m \geq \frac{\theta^* R_0}{2}$,
II) $d_m < \frac{\theta^* R_0}{2}$,

where θ^* has been introduced in Proposition 5.2.

If case I) occurs, let $\bar{d} = \frac{\theta^* R_0}{2(1+\sqrt{1+E^2})}$, $\bar{x} = x_0 - \sqrt{1+E^2}\bar{d}\nu$, where ν denotes the outer unit normal to Ω_1 at x_0. We have that

$$\Delta_{\bar{d}}(\bar{x}) \subset \Omega_1 \cap \Delta_{\frac{\theta^* R_0}{2}}(x_0). \tag{5.17}$$

By (5.8) of Proposition 5.5 and by (5.17) we have

$$\int_0^{T/2} \int_{\Omega_1 \cap \Delta_{\frac{\theta^* R_0}{2}}(x_0)} u_1^2(x,t)dxdt \geq CR_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)}, \tag{5.18}$$

where $C > 0$ depends on λ, Λ, E, M, $\frac{R_0^2}{T}$ and F only. On the other hand, it is evident that

$$d_m \leq CR_0, \tag{5.19}$$

with C depending on E and M only, so that, from (5.16), (5.18), (5.19), and from $d_m \geq \frac{\theta^* R_0}{2}$, we obtain

$$d_m \leq CR_0 \left(\frac{\eta}{R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)}} \right)^{1/3}, \tag{5.20}$$

where C depends on λ, Λ, E, M, $\frac{R_0^2}{T}$ and F only.

If case II) occurs, let us apply (5.3b) of Proposition 5.2 with $r_1 = d_m$, $r_2 = \frac{\theta^* R_0}{2}$, obtaining, by (5.16),

(5.21)

$$\int_0^{T/2} \int_{\Omega_1 \cap \Delta_{\frac{\theta^* R_0}{2}}(x_0)} u_1^2(x,t)dxdt \leq CR_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)} \left(\frac{\eta}{R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)}} \right)^{\bar{\beta}\gamma'},$$

where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only and, moreover,

$$\bar{\beta}\gamma' \geq \frac{C}{\log \frac{R_0}{d_m}}, \tag{5.22}$$

with C depending only on λ, Λ and E, since $d_m < \frac{\theta^* R_0}{2}$. It is not restrictive to assume that $\eta < R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)}$, since otherwise (5.14) becomes trivial. Hence, from (5.18), (5.21) and (5.22) we have

$$\left(\frac{\eta}{R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)}} \right)^{\frac{C}{\log \frac{R_0}{d_m}}} \geq \tilde{C}, \tag{5.23}$$

where C depends on λ, Λ and E only and $\tilde{C}$ depends on λ, Λ, E, M, $\frac{R_0^2}{T}$ and F only. Therefore (5.14) follows.

In order to prove (5.15), let us assume, with no loss of generality, that there exists $y_0 \in \overline{\Omega_1}$ such that $\mathrm{dist}(y_0, \overline{\Omega_2}) = d$. Let us notice that in general y_0 need not belong to $\partial\Omega_1$ (see the example below Definition 5.1). Denoting $h = \mathrm{dist}(y_0, \partial\Omega_1)$, let us distinguish the following three cases:

i) $h \leq \frac{d}{2}$,
ii) $h > \frac{d}{2}$, $h > \frac{d_0}{2}$,
iii) $h > \frac{d}{2}$, $h \leq \frac{d_0}{2}$,

where d_0 is the number introduced in Proposition 5.6.

If case i) occurs, taking $z_0 \in \partial\Omega_1$ such that $|y_0 - z_0| = h$, we have that $\mathrm{dist}(z_0, \overline{\Omega_2}) \geq d - h \geq \frac{d}{2}$, so that $d \leq 2d_m$ and (5.15) follows from (5.14).

If case ii) occurs, let us set

$$d_1 = \min\left\{\frac{d}{2}, \frac{\theta d_0}{4}\right\}, \tag{5.24}$$

where θ, $0 < \theta < 1$, has been introduced in Proposition 5.1. We have that

$$\Delta_{d_1}(y_0) \subset \Omega_1 \setminus \Omega_2, \tag{5.25}$$

$$\Delta_{\frac{\theta d_0}{2}}(y_0) \subset \Omega_1. \tag{5.26}$$

By applying (5.1b) of Proposition 5.1 with $r_1 = d_1$, $r_2 = \frac{\theta d_0}{2}$, and using (5.13) and (5.25), we have

(5.27)

$$\int_0^{T/2} \int_{\Delta_{\frac{\theta d_0}{2}}(y_0)} u_1^2(x,t)\,dx\,dt \leq C R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)} \left(\frac{\eta}{R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)}}\right)^{\bar{\beta}\gamma'},$$

where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only; and, moreover,

$$\bar{\beta}\gamma' \geq \frac{C}{\log \frac{R_0}{d_1}}, \tag{5.28}$$

where C depends on λ, Λ and E only. By (5.26) we may apply (5.8) of Proposition 5.5, obtaining by (5.27), (5.28),

$$d_1 \leq \tilde{C} R_0 \left(\frac{\eta}{R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)}}\right)^K, \tag{5.29}$$

where $\tilde{C} > 0$ and $K > 0$ depend on λ, Λ, E, M, $\frac{R_0^2}{T}$ and F only.

Let $\bar{\eta} = \left(\frac{\theta d_0}{4\tilde{C} R_0}\right)^{\frac{1}{K}} R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)}$. If $\eta < \bar{\eta}$, then $d_1 < \frac{\theta d_0}{4}$, so that $d = 2d_1$ and (5.15) follows from (5.29). If, otherwise, $\eta \geq \bar{\eta}$, then (5.15) follows trivially, like what we did in (5.19).

If case iii) occurs, then $d < d_0$ and Proposition 5.6 applies, so that by (5.12) and (5.14) we again obtain (5.15).

Hence, by Proposition 5.3, we obtain

$$d \leq C R_0 \left(\left|\log\left|\log\left(\frac{R_0^{\frac{n-3}{2}} T^{\frac{1}{2}} \epsilon}{\|g\|_{H^{1/2,1/4}(S_T)}}\right)\right|\right|\right)^{-K}, \tag{5.30}$$

where $C > 0$ and $K > 0$ depend on λ, Λ, E, M, $\frac{R_0^2}{T}$ and F only. Thus we have obtained a stability estimate of log-log type. Next, by (5.30), we can find $\epsilon_0 > 0$, depending only on λ, Λ, E, M, $\frac{R_0^2}{T}$ and F, such that if $\epsilon \leq \epsilon_0$ then $d \leq d_0$. Therefore, by Proposition 5.6, G satisfies the hypotheses of Proposition 5.4. Hence in (5.15) we may replace η by $R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)} \omega\left(\frac{R_0^{\frac{n-3}{2}} T^{\frac{1}{2}} \epsilon}{\|g\|_{H^{1/2,1/4}(S_T)}}\right)$, where ω is given by (5.7) of Proposition 5.4 (a modulus of continuity of log type), and obtain (4.10)−(4.11). $\square$

6. Proofs of Propositions 5.1–5.5

Proof of Proposition 5.1. The thesis follows from (3.1.7b) of Theorem 3.1.1, with $R = \frac{\varrho}{2}$, $r_3 = \bar{\theta}R$, $\theta = (4\max\{\sqrt{2}, \lambda_2\})^{-1}\frac{\bar{\theta}}{2}$, from the trace inequality

$$H^2 \leq \frac{C}{T}\|u\|^2_{H^{2,1}(\Omega\times(0,T))}, \tag{6.1}$$

where C depends on E only, and from the estimate (see [LSU])

$$\|u\|^2_{H^{2,1}(\Omega\times(0,T))} \leq CR_0^3\|g\|^2_{H^{1/2,1/4}(S_T)}, \tag{6.2}$$

where C depends on E and λ only. □

Proof of Proposition 5.2. The thesis follows from (3.2.4b) of Theorem 3.2.1, from the trace inequality

$$\bar{H}^2 \leq \frac{C}{T}\|u\|^2_{H^{2,1}(\Omega\times(0,T))}, \tag{6.3}$$

where C depends on E only, and from (6.2). □

Proof of Proposition 5.3. Let us denote, for $i = 1, 2$,

$$\mathcal{U}_i^r = \{x \in \overline{\Omega_i} \text{ s.t. } \operatorname{dist}(x, A_{R_0}) \leq r\}.$$

It is clear that

$$\mathcal{U}_1^r = \mathcal{U}_2^r = \mathcal{U}^r, \qquad \text{for every } r < R_0.$$

From regularity estimates for solutions to parabolic equations satisfying homogeneous Neumann conditions, we have that $u_i \in C^{1,\alpha}(\overline{\Omega_i \setminus \mathcal{U}^{R_0/8}} \times [0, T])$, $i = 1, 2$, and $u_1 - u_2 \in C^{1,\alpha}(\overline{\Omega_1 \cap \Omega_2} \times [0, T])$ for every $\alpha \in (0, 1)$. Since the value of the exponent α is not relevant to our purposes, let us choose $\alpha = 1/2$, obtaining

$$\|u_i\|_{C^{1,1/2}(\overline{\Omega_i\setminus\mathcal{U}^{R_0/8}}\times[0,T])} \leq CT^{-\frac{1}{2}}R_0^{-\frac{n-3}{2}}\|g\|_{H^{1/2,1/4}(S_T)}, \quad \text{for } i = 1, 2, \tag{6.4}$$

$$\|u_1 - u_2\|_{C^{1,1/2}(\overline{\Omega_1\cap\Omega_2}\times[0,T])} \leq CT^{-\frac{1}{2}}R_0^{-\frac{n-3}{2}}\|g\|_{H^{1/2,1/4}(S_T)}, \tag{6.5}$$

where $C > 0$ depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only. Now, in order to apply Theorem 3.3.1 to $u = u_1 - u_2$ in G, let us estimate $\|u_1 - u_2\|_{H^{3/2,3/4}(\Sigma\times(0,T))}$ in terms of $\|u_1 - u_2\|_{L^2(\Sigma\times(0,T))}$ and of the a priori data. The functions u, u_t and u_{tt} satisfy the equation

$$u_t(x, t) - \operatorname{div}(\kappa(x)\nabla u(x, t)) = 0, \quad \text{in } G \times (0, T],$$

and the homogeneous boundary conditions

$$u = 0, \quad \text{on } \bar{G} \times \{0\},$$

$$\kappa\nabla u \cdot \nu = 0, \quad \text{on } A \times [0, T].$$

Hence we may apply the local boundedness estimate (see [LSU]), obtaining

$$\|u_t\|_{L^\infty(\mathcal{U}^{R_0/8}\times(0,T))} \leq CT^{-\frac{3}{2}}R_0^{-\frac{n-3}{2}}\|g\|_{H^{1/2,1/4}(S_T)}, \tag{6.6a}$$

$$\|u_{tt}\|_{L^\infty(\mathcal{U}^{R_0/8}\times(0,T))} \leq CT^{-\frac{5}{2}}R_0^{-\frac{n-3}{2}}\|g\|_{H^{1/2,1/4}(S_T)}, \tag{6.6b}$$

where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. We may think at $u(\cdot,t)$ as a solution to the elliptic problem

$$\mathrm{div}(\kappa(x)\nabla u(x,t)) = u_t(x,t), \quad \text{in } G,$$

$$\kappa\nabla u(x,t)\cdot\nu = 0, \quad \text{on } A,$$

and, similarly, we may think at $u_t(\cdot,t)$ as a solution to the elliptic problem

$$\mathrm{div}(\kappa(x)\nabla u_t(x,t)) = u_{tt}(x,t), \quad \text{in } G,$$

$$\kappa\nabla u_t(x,t)\cdot\nu = 0, \quad \text{on } A.$$

By L^p regularity estimates (see [GT]), by (6.6), by trace inequalities and by the immersion of $W_p^{2-1/p}$ in $H^{2-1/p}$ for $p>2$, we have

$$\sup_{t\in[0,T]}\left(\|u(\cdot,t)\|_{H^{2-1/p}(\Sigma)} + T\|u_t(\cdot,t)\|_{H^{2-1/p}(\Sigma)}\right) \le CT^{-\frac{1}{2}}R_0\|g\|_{H^{1/2,1/4}(S_T)},$$

for any $p>2$, where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. Therefore

$$\|u\|_{H^{\alpha,\alpha/2}(\Sigma\times(0,T))} \le CR_0\|g\|_{H^{1/2,1/4}(S_T)}, \tag{6.7}$$

with $\alpha = 2-1/p > 3/2$, where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. By interpolation we have

$$\|u\|_{H^{3/2,3/4}(\Sigma\times(0,T))} \le C\|u\|^{1-\theta}_{H^{\alpha,\alpha/2}(\Sigma\times(0,T))}\|u\|^{\theta}_{L^2(\Sigma\times(0,T))}, \tag{6.8}$$

where θ is given by $(1-\theta)\alpha = 3/2$ (see [LioM]). By (4.9), (6.7) and (6.8), choosing $p=4$ we have

$$\|u\|_{H^{3/2,3/4}(\Sigma\times(0,T))} \le CR_0\|g\|_{H^{1/2,1/4}(S_T)}\left(\frac{T^{1/2}R_0^{(n-3)/2}\epsilon}{\|g\|_{H^{1/2,1/4}(S_T)}}\right)^{1/7}, \tag{6.9}$$

where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. By applying Theorem 3.3.1 to u and by (6.2), we have

$$\|u(\cdot,t_0)\|_{L^2(\Delta_{\tilde\theta R_0}(P_2))} \le CR_0^{\frac{n}{2}}\left(\frac{\|g\|_{H^{1/2,1/4}(S_T)}}{T^{1/2}R_0^{(n-3)/2}}\right)^{1-\frac{\bar\gamma}{7}}\epsilon^{\bar\gamma/7}, \tag{6.10}$$

where $P_2 = P_1 - \tilde\theta R_0\nu$, ν denotes the outer unit normal to Ω at P_1, $\bar\gamma$, $0<\bar\gamma<1$, has been introduced in Theorem 3.3.1 and depends on λ and Λ only, $\tilde\theta$, $0<\tilde\theta<1$, depends on λ and E only, $C\ge 1$ depends on λ, Λ, E and $\frac{R_0^2}{T}$ only.

Let us prove (5.5) – (5.6) when $i=1$, the case $i=2$ being analogous. Let $r\le\tilde\theta R_0$. Let V_r be the connected component of $\Omega_{1,r}\cap\Omega_{2,r}$ whose closure contains $\{x\in\Omega_1 \text{ s.t. } \mathrm{dist}(x,\Sigma)=r\}$. We have

$$\Omega_1\setminus G \subset [(\Omega_1\setminus\Omega_{1,r})\setminus G]\cup[\Omega_{1,r}\setminus V_r],$$

$$\partial(\Omega_{1,r}\setminus V_r) = \Gamma_{1,r}\cup\Gamma_{2,r},$$

where $\Gamma_{1,r}$ is the part of boundary contained in $\partial\Omega_{1,r}$ and $\Gamma_{2,r}$ is the part contained in $\partial\Omega_{2,r}\cap\partial V_r$. Let us notice that

$$|\Gamma_{i,r}| \le CR_0^{n-1}, \qquad i=1,2, \tag{6.11a}$$

$$|\Omega_i\setminus\Omega_{i,r}| \le CR_0^{n-1}r, \qquad i=1,2, \tag{6.11b}$$

where C depends on E and M only (see Lemma 2.8 in [AlRos]). We have

$$\int_0^{T/2}\int_{\Omega_1\setminus G} u_1^2 dxdt \leq \int_0^{T/2}\int_{(\Omega_1\setminus\Omega_{1,r})\setminus G} u_1^2 dxdt + \int_0^{T/2}\int_{\Omega_{1,r}\setminus V_r} u_1^2 dxdt. \tag{6.12}$$

By (6.4) and (6.11b) we have

$$\int_0^{T/2}\int_{(\Omega_1\setminus\Omega_{1,r})\setminus G} u_1^2 dxdt \leq C r^{\frac{1}{2}} R_0^{\frac{5}{2}} \|g\|^2_{H^{1/2,1/4}(S_T)}, \tag{6.13}$$

where C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only. By the divergence theorem, we have

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_{1,r}\setminus V_r} u_1^2(x,\tau)dx \\ &\leq \int_0^{\tau}\int_{\Gamma_{1,r}} |(u_1\kappa\nabla u_1\cdot\nu)| dsdt + \int_0^{\tau}\int_{\Gamma_{2,r}} |(u_1\kappa\nabla u_1\cdot\nu)| dsdt, \end{aligned} \tag{6.14}$$

where $\nu(x)$ denotes the unit outer normal to $\Omega_{1,r}$ at x.

Let $x\in\Gamma_{1,r}$. We have that $d(x,\partial\Omega_1)=r$ and, on the other hand, that $d(x,\Sigma)\geq \frac{R_0}{2}>r$, since $x\in\overline{\Omega_{1,r}\setminus V_r}$. Hence, there exists $y\in\partial\Omega_1\setminus\Sigma$ such that $|y-x| = \mathrm{dist}(x,\partial\Omega_1)=r$ and moreover $(\kappa\nabla u_1\cdot\nu)(y,t)=0$ for every $t\in[0,T]$, where $\nu(y)$ is the unit outer normal to Ω_1 at y. Since

$$|\nu(x)-\nu(y)|\leq C\frac{r}{R_0}, \tag{6.15}$$

with C depending only on E, and by (6.4) we have

$$|(\kappa\nabla u_1\cdot\nu)(x,t)|\leq CT^{-\frac{1}{2}}R_0^{-\frac{n}{2}}r^{\frac{1}{2}}\|g\|_{H^{1/2,1/4}(S_T)}, \quad \text{for every } t\in[0,T], \tag{6.16}$$

where C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only.

Similarly, given $x\in\Gamma_{2,r}$, there exists $y\in\partial\Omega_2\setminus\Sigma$ such that $|y-x|=\mathrm{dist}(x,\partial\Omega_2) = r$. Since $(\kappa\nabla u_2\cdot\nu)(y,t)=0$ for every $t\in[0,T]$, we have

$$|(\kappa\nabla u_1\cdot\nu)(x,t)|\leq C\left(T^{-\frac{1}{2}}R_0^{-\frac{n}{2}}r^{\frac{1}{2}}\|g\|_{H^{1/2,1/4}(S_T)}+|\nabla u(x,t)|\right), \quad \forall t\in[0,T], \tag{6.17}$$

where C depends on λ, E, M and $\frac{R_0^2}{T}$ only. From (6.11) – (6.14), (6.16) – (6.17) and (6.4) we have

$$\begin{aligned} &\int_0^{T/2}\int_{\Omega_1\setminus G} u_1^2 dxdt \\ &\leq CR_0^{\frac{5}{2}}\|g\|_{H^{1/2,1/4}(S_T)}\left(r^{\frac{1}{2}}\|g\|_{H^{1/2,1/4}(S_T)}+R_0^{\frac{n+2}{2}}\|\nabla u\|_{L^\infty(\overline{V_r}\times[0,T/2])}\right), \end{aligned} \tag{6.18}$$

where C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only. Let $(\bar{x},\bar{t})\in\overline{V_r}\times[0,T/2]$ be such that $|\nabla u(\bar{x},\bar{t})| = \|\nabla u\|_{L^\infty(\overline{V_r}\times[0,T/2])}$. Since $d(P_2,\Sigma)=\tilde{\theta}R_0$ and $d(P_2,I_1\cup I_2)\geq (1-\tilde{\theta})R_0\geq\tilde{\theta}R_0$, it follows that $d(P_2,\partial G)=\tilde{\theta}R_0\geq r$, so that $P_2\in\overline{V_r}$. Let γ be an arc in $\overline{V_r}$ joining $\bar{x}$ to P_2. Let us define $\{x_i\}$, $i=1,...,s$, as follows: $x_1=P_2$, $x_{i+1}=\gamma(t_i)$, where $t_i=\max\{t \text{ s. t. } |\gamma(t)-x_i|=2\tilde{\theta}r\}$ if $|x_i-\bar{x}|>2\tilde{\theta}r$; otherwise let $i=s$ and stop the process. By construction, the balls $\Delta_{\tilde{\theta}r}(x_i)$ are pairwise disjoint, $|x_{i+1}-x_i|=2\tilde{\theta}r$, for $i=1,...,s-1$, $|x_s-\bar{x}|\leq 2\tilde{\theta}r$. Hence we have $s\leq S\left(\frac{R_0}{r}\right)^n$, with S depending only on λ, E, and M. By iterated application of the three spheres inequality (3.1.7a) to u, with $R = \frac{r}{2}$, $r_1 = \frac{\bar{\theta}}{26\max\{\sqrt{2},\lambda_2\}}r$, $r_2 = 3r_1$, $r_3 = 13\max\{\sqrt{2},\lambda_2\}r_1$, over the chain of balls $\Delta_{r_1}(x_i)$, $i = 1, ..., s$, and by (6.1) – (6.2), we have

(6.19)
$$\|u(\cdot,\bar{t})\|^2_{L^2(\Delta_{r_1}(\bar{x}))} \leq C\left(R_0\left(1+\frac{T^2}{r^4}\right)\|g\|^2_{H^{1/2,1/4}(S_T)}\right)^{1-\bar{\gamma}^s}\|u(\cdot,\bar{t})\|^{\bar{\gamma}^s}_{L^2(\Delta_{r_1}(P_2))},$$

where $\bar{\gamma}$, $0 < \bar{\gamma} < 1$, has been introduced in Theorem 3.3.1 and depends on λ and Λ only, C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. By (6.10), we have

$$\|u(\cdot,\bar{t})\|^2_{L^2(\Delta_{r_1}(\bar{x}))} \leq CR_0\left(1+\frac{T^2}{r^4}\right)^{1-\bar{\gamma}^s}\|g\|^2_{H^{1/2,1/4}(S_T)}\tilde{\epsilon}^{\frac{2}{7}\bar{\gamma}^{s+1}}, \tag{6.20}$$

where

$$\tilde{\epsilon} = \frac{T^{\frac{1}{2}}R_0^{\frac{n-3}{2}}\epsilon}{\|g\|_{H^{1/2,1/4}(S_T)}}, \tag{6.21}$$

and C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. By applying (A.2) to $u(\cdot,\bar{t})$ in $\Delta_{r_1}(\bar{x})$ with $\alpha = 1/2$, by (6.20) and (6.5), we have

$$|\nabla u(\bar{x},\bar{t})| \leq \frac{C}{R_0^{\frac{n+1}{2}}}\left(\frac{R_0}{r}\right)^{\frac{4n+6}{3n+6}}\left(1+\frac{T^2}{r^4}\right)^{\frac{1-\bar{\gamma}^s}{3n+6}}\|g\|_{H^{1/2,1/4}(S_T)}\tilde{\epsilon}^{\frac{2\bar{\gamma}^{s+1}}{7(3n+6)}}, \tag{6.22}$$

where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. From $r \leq \tilde{\theta}R_0$, we have that $r \leq CT^{\frac{1}{2}}$, with C depending on λ, Λ, E and $\frac{R_0^2}{T}$ only. Therefore we can estimate

$$1+\frac{T^2}{r^4} \leq C\left(\frac{R_0}{r}\right)^4,$$

where C depends on λ, Λ, E and $\frac{R_0^2}{T}$ only. By substituting (6.22) in (6.18) and by the above inequality we have

(6.23)
$$\int_0^{T/2}\int_{\Omega_1\setminus G} u_1^2 dxdt \leq CR_0^3\|g\|^2_{H^{1/2,1/4}(S_T)}\left(\left(\frac{r}{R_0}\right)^{\frac{1}{2}} + \left(\frac{R_0}{r}\right)^{\frac{4n+10}{3n+6}}(\tilde{\epsilon})^{\frac{2\bar{\gamma}^{s+1}}{7(3n+6)}}\right),$$

where C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only.

Let $\bar{\mu} = \exp\left\{-\frac{7}{2}(3n+6)\exp\left(\frac{2(S+1)|\log\bar{\gamma}|}{\tilde{\theta}^n}\right)\right\}$, $\tilde{\mu} = \min\{\bar{\mu}, \exp(-[\frac{7}{2}(3n+6)]^2)\}$. We have that $\tilde{\mu} < e^{-1}$, and it depends on λ, Λ, E and M only. It is not restrictive to assume that $\tilde{\epsilon} \leq \tilde{\mu}$, since, otherwise, (5.5) – (5.6) become trivial. Therefore, let $\tilde{\epsilon} \leq \tilde{\mu}$ and let

$$r(\tilde{\epsilon}) = R_0\left(\frac{2(S+1)|\log\bar{\gamma}|}{\log\left|\log\tilde{\epsilon}^{2/(7(3n+6))}\right|}\right)^{1/n}.$$

Since $r(\tilde{\epsilon})$ is increasing in $(0, e^{-1})$ and since $r(\tilde{\mu}) \leq r(\bar{\mu}) = R_0\tilde{\theta}$, inequality (6.23) is applicable when $r = r(\tilde{\epsilon})$ and we obtain

$$\int_0^{T/2}\int_{\Omega_1\setminus G} u_1^2 dxdt \leq CR_0^3\|g\|^2_{H^{1/2,1/4}(S_T)}\left(\log\left|\log\tilde{\epsilon}^{2/(7(3n+6))}\right|\right)^{-1/n}, \tag{6.24}$$

where C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only. Since $\tilde{\epsilon} \leq \exp(-[\frac{7}{2}(3n+6)]^2)$, we have that $\log(\frac{2}{7(3n+6)}) \geq -\frac{1}{2}\log|\log\tilde{\epsilon}|$, so that

$$\log\left|\log\tilde{\epsilon}^{\frac{2}{7(3n+6)}}\right| \geq \frac{1}{2}\log|\log\tilde{\epsilon}|, \tag{6.25}$$

and the thesis follows. □

Proof of Proposition 5.4. Let us prove (5.5) and (5.7) when $i=1$, the case $i=2$ being analogous. By the divergence theorem we have

$$\frac{1}{2}\int_{\Omega_1\setminus G} u_1^2(x,\tau)dx \leq \int_0^\tau\int_{\partial(\Omega_1\setminus G)} u_1(\kappa\nabla u_1\cdot\nu)dsdt, \quad \text{for every } \tau\in[0,T]. \tag{6.26}$$

Since

$$\partial(\Omega_1\setminus G)\subset(\partial\Omega_1\setminus A)\cup((\partial\Omega_2\setminus A)\cap\partial G)$$

and since $\kappa\nabla u_i\cdot\nu=0$ on $(\partial\Omega_i\setminus A)\times[0,T]$, $i=1,2$, we have, by (6.26) and (6.4),

$$\int_0^{T/2}\int_{\Omega_1\setminus G} u_1^2 dxdt \leq CR_0^{\frac{n+7}{2}}\|g\|_{H^{1/2,1/4}(S_T)}\|\nabla u\|_{L^\infty(\partial G\times[0,T/2])}, \tag{6.27}$$

where $u=u_1-u_2$ and C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only.

Let us introduce the following notation.

Given $z\in\mathbb{R}^n$, $\xi\in\mathbb{R}^n$, $|\xi|=1$, $\theta>0$, $r>0$, we shall denote by

$$C(z,\xi,\theta,r)=\{x\in\mathbb{R}^n \text{ s. t. } \frac{(x-z)\cdot\xi}{|x-z|}>\cos\theta, |x-z|<r\}$$

the intersection of the ball $\Delta_r(z)$ with the open cone having vertex z, axis in the direction ξ and width 2θ. Since ∂G is of Lipschitz class with constants r_0, L, for any $z\in\partial G$ there exists $\xi\in\mathbb{R}^n$, $|\xi|=1$, such that $C(z,\xi,\theta,r_0)\subset G$, where $\theta=\arctan\frac{1}{L}$.

Let $(\bar{z},\bar{t})\in\partial G\times[0,T/2]$ be such that $|\nabla u(\bar{z},\bar{t})|=\|\nabla u\|_{L^\infty(\partial G\times[0,T/2])}$. Let

$$\lambda_1=\min\{\frac{r_0}{1+\sin\theta},\frac{r_0}{3\sin\theta},\frac{\tilde{\theta}R_0}{\sin\theta}\},$$

$$\theta_1=\arcsin\left(\frac{\sin\theta}{l}\right),$$

$$w_1=\bar{z}+\lambda_1\xi,$$

$$\rho_1=\lambda_1\sin\theta_1,$$

where $l=13\max\{\sqrt{2},\lambda\}$ and $\tilde{\theta}$ has been introduced in Theorem 3.3.1. We have that $\Delta_{\rho_1}(w_1)\subset C(\bar{z},\xi,\theta_1,r_0)$, $\Delta_{l\rho_1}(w_1)\subset C(\bar{z},\xi,\theta,r_0)\subset G$, so that $w_1\in\overline{G_{l\rho_1}}$. Moreover $P_2\in\overline{G_{l\rho_1}}$ since $l\rho_1\leq\tilde{\theta}R_0$, where P_2 has been introduced in Theorem 3.3.1, and $\overline{G_{l\rho_1}}$ is connected since $l\rho_1\leq\frac{r_0}{3}$. Arguing as in the proof of Proposition 5.3, we obtain, by an iterated application of the three spheres inequality (3.1.7a), by (6.1), (6.2) and (6.10),

$$\|u(\cdot,\bar{t})\|^2_{L^2(\Delta_{\rho_1}(w_1))}\leq CR_0\|g\|^2_{H^{1/2,1/4}(S_T)}\tilde{\epsilon}^{2\beta_1}, \tag{6.28}$$

where β_1, $0<\beta_1<1$, depends on λ, Λ, E, M, L and $\frac{R_0}{r_0}$ only and C depends on λ, Λ, E, $\frac{R_0^2}{T}$ and $\frac{R_0}{r_0}$ only. Let us approach $\bar{z}$, by constructing a sequence of balls contained in $C(\bar{z},\xi,\theta_1,r_0)$. We define, for $k\geq 2$,

$$w_k=\bar{z}+\lambda_k\xi,$$

$$\lambda_k=\chi\lambda_{k-1},$$

$$\rho_k=\chi\rho_{k-1},$$

with

$$\chi=\frac{1-\sin\theta_1}{1+\sin\theta_1}.$$

Hence $\rho_k=\chi^{k-1}\rho_1,\ \lambda_k=\chi^{k-1}\lambda_1,\ \Delta_{\rho_{k+1}}(w_{k+1})\subset\Delta_{3\rho_k}(w_k)\subset\Delta_{l\rho_k}(w_k)\subset C(\bar{z},\xi,\theta,r_0)\subset G$. Denoting

$$d(k)=|w_k-\bar{z}|-\rho_k,$$

we have

$$d(k)=\chi^{k-1}d(1),$$

with

$$d(1)=\lambda_1(1-\sin\theta_1).$$

For any r, $0<r\leq d(1)$, let $k(r)$ be the smallest positive integer such that $d(k)\leq r$, that is,

$$\frac{\left|\log\frac{r}{d(1)}\right|}{|\log\chi|}\leq k(r)-1\leq\frac{\left|\log\frac{r}{d(1)}\right|}{|\log\chi|}+1. \tag{6.29}$$

By an iterated application of the three spheres inequality (3.1.7a) over the chain of balls $\Delta_{\rho_1}(w_1)$,...,$\Delta_{\rho_{k(r)}}(w_{k(r)})$, by (6.1), (6.2) and (6.28), we have

(6.30)

$$\|u(\cdot,\bar{t})\|^2_{L^2(\Delta_{\rho_{k(r)}}(w_{k(r)}))}\leq CR_0\|g\|^2_{H^{1/2,1/4}(S_T)}\left(1+\frac{T^2}{\rho^4_{k(r)}}\right)^{1-\bar{\gamma}^{k(r)-1}}\tilde{\epsilon}^{2\beta_1\bar{\gamma}^{k(r)-1}},$$

where $\bar{\gamma}$, $0<\bar{\gamma}<1$, depends on λ and Λ only and C depends on λ, Λ, E, $\frac{R_0^2}{T}$ and $\frac{R_0}{r_0}$ only. From the definition of $\rho_{k(r)}$ we can estimate

$$1+\frac{T^2}{\rho^4_{k(r)}}\leq\frac{C}{\chi^{4(k(r)-1)}}, \tag{6.31}$$

where C depends on λ, E, L, $\frac{R_0^2}{T}$ and $\frac{R_0}{r_0}$ only. From (6.30), (6.31) and the interpolation inequality (A.2) with $\alpha=1/2$ we obtain

$$\|\nabla u(\cdot,\bar{t})\|_{L^\infty(\Delta_{\rho_{k(r)}}(w_{k(r)}))}\leq\frac{C\|g\|_{H^{1/2,1/4}(S_T)}}{R_0^{\frac{n+1}{2}}}\frac{\tilde{\epsilon}^{\beta_2\bar{\gamma}^{k(r)-1}}}{\chi^{\frac{4n+10}{3n+6}(k(r)-1)}}, \tag{6.32}$$

where $\beta_2 = \frac{\beta_1}{3(n+2)}$ and C depends on λ, Λ, E, L, $\frac{R_0^2}{T}$ and $\frac{R_0}{r_0}$ only. Let us consider the point $z_r = \bar{z} + r\xi$. We have that $z_r \in \Delta_{\rho_{k(r)}}(w_{k(r)})$. From (6.32) and (6.5) we have

$$|\nabla u(\bar{z}, \bar{t})| \leq \frac{C\|g\|_{H^{1/2,1/4}(S_T)}}{R_0^{\frac{n+1}{2}}} \left(\left(\frac{r}{R_0} \right)^{\frac{1}{2}} + \frac{\tilde{\epsilon}^{\beta_2 \bar{\gamma}^{k(r)-1}}}{\chi^{\frac{4n+10}{3n+6}(k(r)-1)}} \right), \tag{6.33}$$

where C depends on λ, Λ, E, M, L, $\frac{R_0^2}{T}$ and $\frac{R_0}{r_0}$ only. Let

$$r(\tilde{\epsilon}) = d(1) \left| \log \tilde{\epsilon}^{\beta_2} \right|^{-B},$$

with

$$B = \frac{|\log \chi|}{2|\log \bar{\gamma}|}.$$

Let $\tilde{\mu} = \exp(-\beta_2^{-1})$. We have that $r(\tilde{\mu}) = d(1)$ and $r(\tilde{\epsilon}) \leq d(1)$ for any $\tilde{\epsilon}$, $0 < \tilde{\epsilon} \leq \tilde{\mu}$. It is not restrictive to assume that $0 < \tilde{\epsilon} \leq \tilde{\mu}$ since, otherwise, (5.5), (5.7) become trivial. Therefore inequality (6.33) is applicable when $r = r(\tilde{\epsilon})$. Recalling (6.27) and (6.29), we obtain

$$\int_0^{T/2} \int_{\Omega_1 \setminus G} u_1^2 dxdt \leq C R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)} \left| \log \tilde{\epsilon}^{\beta_2} \right|^{-B}, \tag{6.34}$$

where C depends on λ, Λ, E, M, L, $\frac{R_0^2}{T}$ and $\frac{R_0}{r_0}$ only. Therefore (5.5) and (5.7) follow. □

Let us precede the proof of Proposition 5.5 with the following lemma, which states a Caccioppoli-type inequality for solutions to parabolic equations vanishing at time $t = 0$.

Lemma 6.1. *Let* $u \in H^{2,1}(\Delta_r \times (0, T)$ *be a solution to* (4.1a)-(4.1b)*, where* κ *satisfies* (4.8a)*. We have*

$$\int_0^\tau \int_{\Delta_s} |\nabla u|^2 dxdt \leq \frac{c\lambda^4}{(r-s)^2} \int_0^\tau \int_{\Delta_r} u^2 dxdt, \tag{6.35}$$

for every $\tau \in [0, T]$ *and for every* $s \in (0, r)$.

Proof of Lemma 6.1. Let $\eta \in C_0^\infty(\Delta_r)$ be such that $\eta \equiv 1$ on Δ_s, $|\nabla \eta| \leq \frac{c}{r-s}$. By using as test function $\varphi = \eta^2 u$, we get

$$\int_0^\tau \int_{\Delta_r} \eta^2 \kappa \nabla u \cdot \nabla u dxdt + \int_0^\tau \int_{\Delta_r} 2\eta u \kappa \nabla u \cdot \nabla \eta dxdt = \frac{1}{2} \int_{\Delta_r} \eta^2 u^2(x, \tau) dx \leq 0.$$

Hence

$$\lambda^{-1} \int_0^\tau \int_{\Delta_r} \eta^2(x) |\nabla u|^2 dxdt$$

$$\leq 2\lambda \left(\int_0^\tau \int_{\Delta_r} \eta^2 |\nabla u|^2 dxdt \right)^{1/2} \left(\int_0^\tau \int_{\Delta_r} u^2 |\nabla \eta|^2 dxdt \right)^{1/2}.$$

Finally, we have

$$\int_0^\tau \int_{\Delta_s} |\nabla u|^2 dxdt \leq \int_0^\tau \int_{\Delta_r} \eta^2 |\nabla u|^2 dxdt \leq \frac{4\lambda^4 c^2}{(r-s)^2} \int_0^\tau \int_{\Delta_r} u^2 dxdt,$$

and the thesis follows. □

Proof of Proposition 5.5. Arguing similarly to the proof of Proposition 4.3 in [AlBRV], we can prove the following trace inequality. Let $l = 13\max\{\sqrt{2}, \lambda\}$. For every $d \leq \frac{R_0}{8l\sqrt{1+E^2}}$ and for every $t \in [0,T]$ we have

$$\int_{\partial\Omega} u_{x_i}^2(x,t)ds \leq C\left(d\int_{\Omega} |\nabla u_{x_i}(x,t)|^2 dx + \frac{1}{d}\int_{\Omega_{ld}} u_{x_i}^2(x,t)dx\right), \tag{6.36}$$

and, integrating (6.36) over $[0,T/2]$,

$$\int_0^{T/2}\int_{\partial\Omega} u_{x_i}^2 dsdt \leq C\left(d\int_0^{T/2}\int_{\Omega} |D^2u|^2 dxdt + \frac{1}{d}\int_0^{T/2}\int_{\Omega_{ld}} u_{x_i}^2 dxdt\right), \tag{6.37}$$

for $i = 1, ..., n$, where C depends on E and M only. Let $\rho > 0$ and $x_0 \in \Omega_\rho$. Let us denote

$$\epsilon^2 = \int_0^{T/2}\int_{\Delta_\rho(x_0)} u^2 dxdt.$$

Let $d \leq d_0$, with $d_0 = \min\{\frac{R_0}{8l\sqrt{1+E^2}}, \frac{\bar{\theta}\rho}{2l}, \frac{R_0}{3l}\}$, where $\bar{\theta}$ has been introduced in Proposition 3.5. Let us notice that $\frac{R_0}{d_0}$ depends on λ, Λ, E, $\frac{R_0^2}{T}$ and $\frac{R_0}{\rho}$ only. Moreover $x_0 \in \Omega_{ld}$ and Ω_{ld} is connected. Let $x \in \Omega_{ld}$. By iterating the three cylinders inequality (3.1.7b) with $r_1 = d$, $r_2 = 3d$, $r_3 = ld$ over a chain of cylinders $\Delta_d(x_k) \times [0,T/2]$, with x_k, $k = 1, ..., N_d$, points of a path in Ω_{ld} joining x to x_0, and by (6.1), we obtain

$$\int_0^{T/2}\int_{\Delta_d(x)} u^2 dxdt \leq C\left(\left(1 + \frac{T^2}{d^4}\right)\|u\|_{H^{2,1}(\Omega\times(0,T))}^2\right)^{1-\bar{\gamma}^{N_d}} \epsilon^{2\bar{\gamma}^{N_d}}, \tag{6.38}$$

where $\bar{\gamma}$, $0 < \bar{\gamma} < 1$, depends on λ and Λ only, C depends on λ, Λ and $\frac{R_0^2}{T}$ only, and $N_d \leq \frac{MR_0^n}{\omega_n d^n}$. Let us cover Ω_{ld} with internally nonoverlapping closed cubes $\{Q_j\}_{j=1,...N_d'}$, of side $d/(2\sqrt{n})$. For every $j = 1, ..., N_d'$, let $x^j \in \Omega_{ld} \cap Q_j$, so that $Q_j \subset \Delta_{d/2}(x_j)$. By Lemma 6.1 we have, for every $i = 1, ..., n$,

$$\int_0^{T/2}\int_{\Omega_{ld}} u_{x_i}^2 dxdt \leq \sum_{j=1}^{N_d'}\int_0^{T/2}\int_{Q_j} u_{x_i}^2 dxdt$$

$$\leq \sum_{j=1}^{N_d'}\int_0^{T/2}\int_{\Delta_{d/2}(x_j)} u_{x_i}^2 dxdt \leq \frac{C}{d^2}N_d'\int_0^{T/2}\int_{\Delta_{d/2}(x_j)} u^2 dxdt, \tag{6.39}$$

where $N_d' \leq C\frac{R_0^n}{d^n}$, with C depending only on M. Let $\alpha = \frac{\epsilon^2}{e^2\|u\|_{H^{2,1}(\Omega\times(0,T))}^2}$. From (6.37) – (6.39) we have

$$\int_0^{T/2}\int_{\partial\Omega} |\nabla u|^2 dsdt \leq \frac{C}{R_0^3}\|u\|_{H^{2,1}(\Omega\times(0,T))}^2\left(\frac{d}{R_0} + \left(\frac{R_0}{d}\right)^{n+7}\alpha^{\bar{\gamma}^{N_d}}\right), \tag{6.40}$$

where C depends on λ, Λ, M, E and $\frac{R_0^2}{T}$ only. Let

$$\alpha_0 = \exp\left(-\exp\left(\frac{2M|\log\bar{\gamma}|}{\omega_n}\left(\frac{R_0}{d_0}\right)^n\right)\right) < 1.$$

We have that α_0 depends on λ, Λ, E, M, $\frac{R_0^2}{T}$ and $\frac{R_0}{\rho}$ only. If $\alpha \leq \alpha_0$ we choose $d = d_\alpha$, where

$$d_\alpha = R_0 \left(\frac{2M|\log \bar{\gamma}|}{\omega_n \log|\log \alpha|} \right)^{1/n}.$$

Then $d_\alpha \leq d(\alpha_0) = d_0$, and from (6.40) we have

$$\int_0^{T/2} \int_{\partial\Omega} |\nabla u|^2 ds dt \leq \frac{C}{R_0^3} \|u\|^2_{H^{2,1}(\Omega\times(0,T))} \left(\log|\log \alpha|\right)^{-1/n}, \tag{6.41}$$

where C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only. If $\alpha > \alpha_0$, from (6.40), using the fact that $\alpha \leq 1$, and replacing d by d_0, we obtain

$$\int_0^{T/2} \int_{\partial\Omega} |\nabla u|^2 ds dt \leq \frac{C}{R_0^3} \|u\|^2_{H^{2,1}(\Omega\times(0,T))} \left(\frac{R_0}{d_0}\right)^{n+7} \frac{\alpha}{\alpha_0}, \tag{6.42}$$

where C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only.

Since $\alpha \leq e^{-2}$ we have $\alpha(\log|\log\alpha|)^{1/n} \leq e^{-2}(\log 2)^{1/n}$, so that from (6.41) – (6.42) we have

$$\int_0^{T/2} \int_{\partial\Omega} g^2 ds dt \leq \frac{C}{\alpha_0 R_0^3} \|u\|^2_{H^{2,1}(\Omega\times(0,T))} \left(\frac{R_0}{d_0}\right)^{n+7} (\log|\log \alpha|)^{-1/n}, \tag{6.43}$$

where C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only. From (6.43) and from

$$C_1 R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)} \leq \|u\|^2_{H^{2,1}(\Omega\times(0,T))} \leq C_2 R_0^3 \|g\|^2_{H^{1/2,1/4}(S_T)}, \tag{6.44}$$

where C_1, C_2 depend on λ, E only, we have

$$\epsilon^2 \geq e^2 R_0^3 \exp\left(-\exp\left(\frac{C}{\alpha_0} \left(\frac{R_0}{d_0}\right)^{n+7} F^2 \right)^n \right) \|g\|^2_{H^{1/2,1/4}(S_T)}, \tag{6.45}$$

where C depends on λ, Λ, E, M and $\frac{R_0^2}{T}$ only. Therefore (5.8) follows with the stated dependence. □

Appendix A. Interpolation and Trace Inequalities

Given an interval I in $\mathbb{R}$, we have

$$\|f\|_{L^\infty(I)} \leq c \left(|I| \|f'\|^2_{L^2(I)} + \frac{1}{|I|} \|f\|^2_{L^2(I)} \right)^{1/2}, \tag{A.1}$$

where $|I|$ denotes the length of the interval I.

Given a function v of class $C^{1,1}$ in $\Delta_\rho \subset \mathbb{R}^n$, for every $\alpha \in (0,1]$ we have

$$\|\nabla v\|_{L^\infty(\Delta_\rho)} \leq \frac{c}{\rho^{1+\frac{n\alpha}{(n+2)(1+\alpha)}}} \|v\|_{C^{1,\alpha}}^{\frac{n\alpha+n+2}{(n+2)(1+\alpha)}} \|v\|_{L^2}^{\frac{2\alpha}{(n+2)(1+\alpha)}}. \tag{A.2}$$

$$\begin{aligned} &\left(\frac{1}{\rho^{n-2}} \int_{\Delta_\rho} |\nabla f|^2 dx \right)^{1/2} \\ &\leq C \left(\left(\rho^{1+\alpha} |\nabla f|_{\alpha,\rho'}\right)^{\frac{1}{1+\alpha}} \left(\frac{1}{\rho^n} \int_{\Delta_\rho} f^2 dx \right)^{\frac{\alpha}{2(1+\alpha)}} + \left(\frac{1}{\rho^n} \int_{\Delta_\rho} f^2 dx \right)^{1/2} \right), \end{aligned} \tag{A.3}$$

where $\rho < \rho' < 2\rho$, $0 < \alpha < 1$ and C depends on α only.

For every $\rho < r$, we have

(A.4)
$$\int_{\Delta_\rho} |F(x,0)|^2 dx \le c\left(\frac{r}{r^2-\rho^2}\int_{B_r^+} |F(x,y)|^2 dxdy + r\int_{B_r^+} |F_y(x,y)|^2 dxdy\right),$$

(A.5)
$$\int_{B_r^+} |F(x,y)|^2 dxdy \le c\left(r\int_{\Delta_r} |F(x,0)|^2 dx + r^2\int_{B_r^+} |F_y(x,y)|^2 dxdy\right).$$

Acknowledgment

The authors wish to thank Giovanni Alessandrini for reading the first draft of this paper and for useful suggestions.

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